Université Pierre et Marie Curie

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Lie Fu

Sous-structures de Hodge, anneaux de Chow et action de certains automorphismes

---

dirigée par Claire Voisin

Soutenue le 3 octobre 2013 devant le jury composé de :

M. Olivier Debarre	École Normale Supérieure	Examinateur
M. Bruno Kahn	Institut de Mathématiques de Jussieu	Examinateur
M. Laurent Manivel	Université de Grenoble I	Examinateur
M. Christian Peskine	Université de Paris VI	Examinateur
M. Chris Peters	Université de Grenoble I	Examinateur
M. Christoph Sorger	Université de Nantes	Rapporteur
Mme Claire Voisin	École Polytechnique	Directrice

Rapporteur absent lors de la soutenance :

M. Kieran O'Grady Università di Roma I

Institut de Mathématiques de Jussieu
4 Place Jussieu
75 005 Paris

École doctorale Paris centre Case 188
4 place Jussieu
75 252 Paris cedex 05

獻給我親愛的父母

子曰
學而不思則罔
思而不學則殆

論語 ● 為政

Remerciements

Je tiens d'abord à exprimer chaleureusement ma gratitude à ma directrice de thèse Claire Voisin. Cette thèse n'aurait jamais vu le jour sans les suggestions pertinentes qu'elle m'a faites, sa générosité à partager ses connaissances et idées mathématiques, la patience qu'elle a montrée pendant de nombreuses heures de discussions et annotation des versions préliminaires de chacun de ces chapitres. C'est une grande chance et un honneur de pouvoir faire mes études et travaux de recherche en mathématiques avec elle.

Je remercie ensuite Christoph Sorger et Kieran O'Grady d'avoir accepté d'être rapporteurs de cette thèse. Leurs questions et remarques m'ont été précieuses. Je remercie sincèrement Olivier Debarre, Bruno Kahn, Laurent Manivel, Christian Peskine et Chris Peters de m'avoir fait l'honneur de faire partie du jury.

Je tiens à remercier particulièrement Olivier Debarre. Son excellent cours 'une introduction à la théorie de Mori' m'a orienté vers la géométrie algébrique complexe. C'est avec son encouragement que je suis devenu moniteur au département de mathématiques de l'ENS, que j'ai dirigé les TD du cours de topologie algébrique et que j'ai fait du tutorat pour les élèves internationaux. Je lui suis très reconnaissant aussi pour son soutien constant pour toute ma mission mathématique.

Je remercie Chiara Camere, Philippe Eyssidieux, Ben Moonen et Olivier Wittenberg pour des discussions enrichissantes. Je suis reconnaissant à Junyan Cao de m'avoir invité à donner un exposé à Grenoble.

Je profite de l'occasion pour remercier mes professeurs : David Harari et Jean-François Dat pour m'avoir introduit aux bases de la géométrie algébrique, Olivier Debarre qui m'a appris beaucoup de techniques pratiques en géométrie, Bruno Kahn pour m'avoir dirigé lors de mon mémoire de Master et pour m'avoir orienté vers l'étude des cycles algébriques, Bruno Klingler pour ses cours d'approfondissement qui m'ont présenté une perspective attrayante et unifiée de la géométrie algébrique.

Je remercie toute l'équipe du Département de Mathématiques et Applications de l'École Normale Supérieure où j'ai effectué la plupart de mon travail de thèse. La magnifique ambiance 'de famille' sur les toits m'a été précieuse. J'ai eu des nombreuse conversations agréables pendant les déjeuners et les cafés avec Omid Amini, Viviane Baladi, Olivier Benoist, Grégory Ginot, Philippe Gille, Henri Guenancia, Harald Helfgott, Martin Hils, Ting-Yu Lee, Tony Ly, Silvain Rideau, Oliver Taïbi, Daniel Schnellmann, Sobhan Seyfaddini, Antoine Touzé et Olivier Wittenberg. Je remercie particulièrement mon cobureau Antoine Touzé avec qui j'ai eu le plaisir de collaborer pour les TDs de son cours topologie algébrique à l'ENS. Je remercie Olivier Biquard pour son soutien, son encouragement et ses conseils pour mon enseignement. Je remercie Zaïna et Laurence pour leur aide administrative.

Je voudrais accorder une place à ma 'famille' mathématique, ma grande soeur Anna Otwinowska, mon grand frère François Charles et mon petit frère Lin Hsueh-Yung pour des discussions inspirantes.

Je voudrais remercier mes collègues au site Jussieu et ailleurs : Dimitri Ara, Jonathan Chiche, Charles De Clerques, Ruben Dashyan, Clément Dupont, Anne Giralt, Liana Heuberger, François Lê, Thibaud Lemanissier, Florent Martin, François Petit, Daniele Turchetti, Jingzhi Yan. Plus particulièrement, je remercie Giuseppe Ancona, Yohan Brunebarbe, Jeremy Daniel, Dragos Fratila et Javier Fresán pour les discussions qui témoignent de nos progrès communs pas à pas en mathématiques.

Je remercie tous mes amis chinois avec qui j'ai partagé mathématiques et bon moments : Junyan Cao, Huan Chen, Linxiao Chen, Yuanmi Chen, Bo Gu, Weikun He, Kai Jiang, Fangzhou Jin, Ben-ben Liao, Gang Li, Jie Lin, Shen Lin, Chunhui Liu, Linyuan Liu, Shi-nan Liu, Tianhan Liu, Ying Liu, Yang Lu, Zhenkai Lu, Li Ma, Yue Ma, Wenhao Ou, Yanqi Qiu, Peng Shan, Getao Shi, Ruxi Shi, Fei Sun, Zhe Tong, Li Wang, Haoran Wang, Hua Wang, Menglin Wang, Lu Wang, Binbin Xu, Daxin Xu, Disheng Xu, Tie Xu, Yingjie Xu, Ze Xu, Jianfeng Yao, Lizao Ye, Qizheng Yin, Yue Yu, Shun Zhang, Xiaoxing Zhang, Ye-ping Zhang, Zhiyuan Zhang, Xingxing Zhou et Jialin Zhu. Je remercie Ziyang Gao, Yong Hu et Yongqi Liang, en particulier pour leur collaboration et soutien pour l'organisation du Séminaire MathJeunes. Finalement, un remerciement spécial à mes deux amis Zhi Jiang et Junyi Xie : après tout ce que nous avons passé ensemble, tout ce que je peux dire est simplement trop faible pour notre amitié.

J'exprime ma reconnaissance du fond du coeur à mes parents pour leur soutien constant. Enfin, toute ma gratitude sincère à Peiming Yuan pour le bonheur que l'on partage.

Résumé

Cette thèse se compose de trois chapitres. Dans Chapitre 1, en supposant la conjecture standard de Lefschetz, on démontre la conjecture de Hodge généralisée pour une sous-structure de Hodge de convieau 1 qui est le noyau du cup-produit avec une classe de cohomologie grosse. Dans Chapitre 2, nous établissons une décomposition de la petite diagonale de $X \times X \times X$ pour une intersection complète de type Calabi-Yau X dans un espace projectif. Comme une conséquence, on déduit une propriété de dégénérescence pour le produit d'intersection dans son anneau de Chow des deux cycles algébriques de dimensions complémentaires et strictement positives. Dans Chapitre 3, on démontre qu'un automorphisme symplectique polarisé de la variété des droites d'une hypersurface cubique de dimension 4 agit trivialement sur son groupe de Chow des 0-cycles, comme prédit par la conjecture de Bloch généralisée.

Mots-clefs

Cycles algébriques ; anneaux de Chow ; structures de Hodge ; coniveau ; conjecture standard ; conjecture de Hodge (généralisée) ; conjecture de Bloch (généralisée) ; spread ; décomposition de la (petite) diagonale ; variétés de type Calabi-Yau ; variétés hyperkählériennes ; automorphismes symplectiques ; variété des droites ; motifs ; conjecture de Bloch-Beilinson-Murre.

Sub-Hodge structures, Chow rings and action of certain automorphisms

Abstract

This Thesis consists of three chapters. In Chapter 1, admitting the Lefschetz standard conjecture, we prove the generalized Hodge conjecture for the kernel of the cup product with a big cohomology class, which is a sub-Hodge structure of coniveau 1. In Chapter 2, we establish a decomposition of the small diagonal of $X \times X \times X$ for a Calabi-Yau complete intersection X in a projective space. As a corollary, we prove that the intersection product, modulo rational equivalence, of algebraic cycles with positive and complementary dimensions, is as degenerated as possible. In Chapter 3, we prove that a polarized symplectic automorphism of the Fano variety of lines of a cubic fourfold acts as identity on its Chow group of 0-cycles, as is predicted by the generalized Bloch conjecture.

Keywords

Algebraic cycles; Chow rings; Hodge structures; coniveau; standard conjecture; (generalized) Hodge conjecture; (generalized) Bloch conjecture; spread; decomposition of the (small) diagonal; Calabi-Yau varieties; hyper-Kähler varieties; symplectic automorphisms; Fano variety of lines; motives; Bloch-Beilinson-Murre conjecture.

Table des matières

Introduction

Étant donnée une variété projective lisse complexe, on s'intéresse d'une part l'étude de ses sous-variétés et les intersections entre elles, d'autre part la topologie, plus précisément sa cohomologie de Betti, de cette variété. La première correspond à l'étude de son anneau de Chow ; la cohomologie est enrichie par la donnée des structures de Hodge. Le but ultime serait de comprendre les relations entre ces deux objets.

0.1 Notions de base

On introduit d'abord les deux invariants algébriques naturellement associés à une variété projective lisse complexe, à savoir, l'anneau de Chow et l'anneau de cohomologie de Betti muni de sa structure de Hodge.

0.1.1 Cycles algébriques et anneaux de Chow

Soit X une variété algébrique intègre, séparée, de type fini sur un corps. Un *cycle* de dimension i est par définition une somme formelle finie de sous-variétés intègres de dimension i dans X. Le groupe abélien libre engendré par les cycles de dimension i est noté $\mathcal{Z}_i(X)$; on pose aussi $\mathcal{Z}^i(X) := \mathcal{Z}_{\dim X-i}(X)$.

On définit ensuit $\mathcal{Z}_r(X)_{\mathrm{rat}}$ comme le sous groupe abélien de $\mathcal{Z}_r(X)$ engendré par les éléments de la forme $W(0) - W(1)$, où W est une sous variété intègre de $X \times \mathbf{P}^1$ plate sur $\mathbf{P}^1$, et $W(0), W(1)$ est la fibre de $W \to \mathbf{P}^1$ au dessus du point 0, 1 respectivement. Deux cycles algébriques de codimension i sont dit *rationnellement équivalents*, si leur différence est contenue dans le sous groupe $\mathcal{Z}_i(X)_{\mathrm{rat}}$. De même, $\mathcal{Z}^i(X)_{\mathrm{rat}} := \mathcal{Z}_{\dim X-i}(X)_{\mathrm{rat}}$.

Définition 0.1.1. Le *groupe de Chow* de codimension i de X

$$\mathrm{CH}^i(X) := \mathcal{Z}^i(X)/\mathcal{Z}^i(X)_{\mathrm{rat}}$$

est par définition le quotient du groupe des cycles algébriques de codimension i modulo l'équivalence rationnelle.

Si X est de plus projective et lisse, alors par le lemme de 'déplacement'

$$\mathrm{CH}^*(X) := \bigoplus_{i=0}^{\dim X} \mathrm{CH}^i(X)$$

admet un produit d'intersection $\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)$, qui fait de $\mathrm{CH}^*(X)$ un anneau unitaire gradué commutatif et associatif, appelé *l'anneau de Chow* de X.

Exemple 0.1.2. Si X est une variété normale à singularités localement factorielles, alors $\mathrm{CH}^1(X)$ n'est rien d'autre que le groupe de Picard $\mathrm{Pic}(X)$ qui classifie les fibrés en droite sur X.

Exemple 0.1.3. Soit C une courbe lisse et propre sur un corps algébriquement clos, alors $\mathrm{CH}_0(C)$ est situé dans une suite exact courte scindée :

$$0 \to \mathrm{Jac}(C) \to \mathrm{CH}_0(C) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

0.1.2 Structures de Hodge

Soit X une variété complexe de dimension n, munie d'une métrique hermitienne h. On note la 2-forme $\omega := -\Im(h)$, où $\Im(h)$ est la partie imaginaire de h. On dit que X est *kählérienne*, si $d\omega = 0$.

Proposition 0.1.4. *La classe des variétés kählériennes compactes est stable par sous variété fermée, produit, fibré projectif, éclatement le long d'une sous variété et petite déformation.*

En particulier, grâce à la métrique de Fubini-Study sur espaces projectifs, toute variété projective lisse définie sur $\mathbf{C}$ admet une métrique kählérienne. La notion de 'structure de Hodge' provient du théorème fondamental suivant dû à Hodge :

Théorème 0.1.5 (Décomposition de Hodge)**.** *Soit X une variété kählérienne compacte de dimension n, alors pour tout entier positif $k \leq 2n$, on a une décomposition*

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

où $H^{p,q}(X)$ est consisté des classes représentables par des (p,q)−formes.
On a évidement $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Définition 0.1.6. Une *structure de Hodge* (*resp. rationnelle*) (pure effective) de poids k est un groupe abélien sans torsion de type fini (*resp.* $\mathbf{Q}$-espace vectoriel de dimension finie) L muni d'une décomposition sur son complexifié :

$$L \otimes \mathbf{C} = \bigoplus_{\substack{p+q=k \\ p,q \in \mathbf{N}}} L^{p,q}$$

telle que $L^{p,q} = \overline{L^{q,p}}$.

En particulier, le k-ième groupe de cohomologie singulière (modulo sa torsion) d'une variété kählérienne compacte est une structure de Hodge de poids k. Une application linéaire entre deux structures de Hodge est un *morphisme de structures de Hodge*, si son complexifié préserve les décompositions. La catégorie des structures de Hodge rationnelles est abélienne.

Du point de vue de la théorie de Hodge, ce qui distingue la classe des variétés projectives lisses des variétés kählériennes compactes est la structure supplémentaire de 'polarisation' :

Définition 0.1.7. Soit L une $\mathbf{Q}$-structure de Hodge de poids k. Une *polarisation* sur L est une forme bilinéaire $(-1)^k$-symétrique :

$$(-,-) : L \otimes_{\mathbf{Q}} L \to \mathbf{Q},$$

vérifiant les propriétés suivantes :

- $(x, y) = 0$, pour $x \in H^{p,q}, y \in H^{p',q'}$ et $(p,q) \neq (q',p')$;
- $i^{p-q}(x, \overline{x}) > 0$, pour $0 \neq x \in H^{p,q}$.

En particulier, par les relations bilinéaires de Hodge-Riemann,

$$(x,y) := (-1)^{\frac{k(k-1)}{2}} \int_X c_1(\mathscr{O}_X(1))^{n-k} \cup x \cup y$$

fournit une polarisation sur la cohomologie primitive

$$H^k(X,\mathbf{Q})_{prim} := \ker\left(\cup c_1(\mathscr{O}_X(1))^{n-k+1} : H^k(X,\mathbf{Q}) \to H^{2n-k+2}(X,\mathbf{Q})\right)$$

pour $k \leq n$, où n est la dimension de X et $\mathscr{O}_X(1)$ est un fibré en droite ample sur X.

Remarque 0.1.8. Soient L une $\mathbf{Q}$-structure de Hodge munie d'une polarisation $(-,-)$, L' une sous $\mathbf{Q}$-structure de Hodge de L. Alors $(-,-)|_{L'}$ est une polarisation, et $L = L' \oplus L'^{\perp}$ est une décomposition orthogonale. En particulier, la sous catégorie pleine des Q-structures de Hodge polarisable est une catégorie abélienne *semi-simple*.

0.1.3 Conjecture de Hodge

Soit X une variété projective lisse complexe. Pour comparer son anneau de Chow $\mathrm{CH}^*(X)$ et son anneau de cohomologie de Betti $H^*(X,\mathbf{Z})$, on considère *l'application classe de cycle* qui associe à chaque sous-variété intègre $Y \subset X$ le dual de Poincaré du push-forward de la classe fondamentale d'une désingularisation $\widetilde{Y}$ de Y :

$$\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X,\mathbf{Z}).$$

Exemple 0.1.9 (La première classe de Chern)**.** La suite exacte longue associée à la suite exacte courte exponentielle

$$0 \to \mathbf{Z} \xrightarrow{2\pi i} O_X \xrightarrow{\exp} O_X^* \to 0$$

induit

$$H^1(X,O_X^*) \xrightarrow{c_1} H^2(X,\mathbf{Z}),$$

où $H^1(X,O_X^*) = \mathrm{Pic}(X) = \mathrm{CH}^1(X)$ s'identifie au groupe de Picard qui paramètre les classes d'isomorphisme de fibré en droite. On peut donc identifier le morphisme de première classe de Chern ci-dessus avec l'application de classe de cycle :

$$\begin{array}{rcl} c_1 : \mathrm{Pic}(X) \simeq \mathrm{CH}^1(X) & \xrightarrow{\mathrm{cl}} & H^2(X;\mathbf{Z}) \\ L \simeq O_X(D) & \mapsto & c_1(L) = \mathrm{cl}(D). \end{array}$$

L'application injective $\mathbf{Z} \xrightarrow{2\pi i} \mathbf{C}$ induit un *morphisme de changement de coefficient*

$$\phi : H^{2k}(X,\mathbf{Z}) \to H^{2k}(X,\mathbf{C}),$$

par la définition de l'application de classe, l'image du composé $\phi \circ \mathrm{cl}$ sera bien dans le facteur $H^{k,k}(X)$. Autrement-dit, la classe de cohomologie d'un cycle algébrique de codimension k est une classe entière de type (k,k) dans la décomposition de Hodge :

$$\mathrm{Im}\left(\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X,\mathbf{Z})\right) \subset Hdg^{2k}(X,\mathbf{Z}),$$

où $Hdg^{2k}(X,\mathbf{Z}) := \phi^{-1}(H^{k,k}) \subset H^{2k}(X,\mathbf{Z})$ est le sous groupe des *classes de Hodge entières*. En particulier, toute classe de torsion dans $H^{2k}(X,\mathbf{Z})$ est une classe de Hodge entière.

La conjecture originale de Hodge est la réciproque :

Conjecture 0.1.10 (Conjecture de Hodge Entière)**.** *Soit X une variété projective lisse sur* $\mathbf{C}$ *de dimension n. Toute classe de Hodge entière $\alpha \in Hdg^{2k}(X, \mathbf{Z})$ provient d'un cycle algébrique :*

$$\mathrm{Im}\left(\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X, \mathbf{Z})\right) = Hdg^{2k}(X, \mathbf{Z}).$$

Remarques 0.1.11. (1) Cette conjecture est triviale pour $k = 0$ ou n. D'après le théorème de Lefschetz sur les classes $(1, 1)$, la conjecture de Hodge entière est vraie pour des classes de Hodge de degré 2.
(2) La conjecture est vraie pour les variétés admettant une décomposition cellulaire, par exemple, espaces projectifs, Grassmanniennes, *etc.*
(3) Le premier contre-exemple à la conjecture de Hodge entière a été fourni par Atiyah-Hirzebruch [7]. Ils ont utilisé la K-théorie pour démontrer qu'une certaine classe de Hodge de torsion n'est pas la classe d'un cycle algébrique. Totaro [63] réinterprète ce résultat dans le cadre de la théorie de cobordisme algébrique, et obtient plus de classes de torsion qui sont des contre-exemples à la conjecture de Hodge entière.
(4) Les classes de Hodge non-torsion posent encore des problèmes. Kollár construit par une méthode de déformation une telle classe de Hodge entière non-torsion qui ne peut pas être algébrique.
(5) Pour les variétés de dimension 3, Voisin [70] montre la conjecture de Hodge entière en dimension 3 pour les variétés uniréglées ou de Calabi-Yau.
(6) Il est facile de construire des contre-exemples pour variété uniréglée en dimension au moins 4 en prenant le produit de $\mathbf{P}^1$ avec l'exemple de Kollár. On pourrait renforcer l'hypothèse en 'rationnellement connexe' (ou même 'unirationnelle'), cependant, dans [61], Soulé et Voisin construisent à partir de l'exemple de Kollár, des variétés rationnelles avec des classes de Hodge entières non algébriques. D'ailleurs, Colliot-Thélène et Voisin [24] construisent des variétés rationnellement connexes (ou même unirationnelles) de dimension au moins 6, comme contre-exemples à la conjecture de Hodge entière.

En passant aux coefficients rationnels, on évite tous les contre-exemples ci-dessus. On note le sous-$\mathbf{Q}$-espace des *classes de Hodge rationnelles* :$Hdg^{2k}(X, \mathbf{Q}) := H^{2k}(X, \mathbf{Q}) \cap H^{k,k}(X) \subset H^{2k}(X, \mathbf{C})$.

Conjecture 0.1.12 (Conjecture de Hodge)**.** *Soit X une variété projective lisse sur* $\mathbf{C}$ *de dimension n. Toute classe de Hodge rationnelle $\alpha \in Hdg^{2k}(X, \mathbf{Q})$ provient d'un* $\mathbf{Q}$*−cycle algébrique : il existe* $z \in \mathrm{CH}^k(X)_{\mathbf{Q}}$ *telle que* $\mathrm{cl}(z) = \alpha$.

Remarques 0.1.13. (1) Grâce au théorème de Lefschetz difficile, la conjecture de Hodge pour le degré $2k$ ($k \leq n$) implique celle pour le degré $2(n - k)$. En particulier, la conjecture vaut pour $k = 0, 1, n - 1, n$.
(2) La conjecture de Hodge vaut pour les variétés admettant une décomposition cellulaire, par exemple, G/P où G est un groupe linéaire algébrique et P est un sous-groupe parabolique.
(3) Voisin [66] montre que la conjecture de Hodge ne peut pas se généraliser à toutes les variétés kählériennes, même les classes de Chern des faisceaux cohérents ne suffisent pas pour engendrer toutes les classes de Hodge rationnelles.
(4) La conjecture est démontrée pour les hypersurfaces cubiques lisses de dimension 4 par Zucker [78].

0.2 Relations entre la topologie et les cycles algébriques

On commence étudier les relations entre l'anneau de cohomologie et l'anneau de Chow en résumant quelques résultats généraux en cette section. Dans cette section, X est toujours une variété projective lisse complexe.

0.2.1 Coniveau géométrique et de Hodge, conjecture de Hodge généralisée

Dans [35], Grothendieck introduit la *filtration de coniveau géométrique* sur la cohomologie $H^k(X,\mathbf{Q})$:

$$N^cH^k(X,\mathbf{Q}) := \sum_{\substack{Y\subset X \text{ fermé} \\ \mathrm{codim}_X(Y)\geq c}} \ker\left(H^k(X,\mathbf{Q}) \to H^k(X\backslash Y,\mathbf{Q})\right).$$

D'une manière équivalente, $N^cH^k(X,\mathbf{Q})$ est constitué des classes *supportées* sur un sous-ensemble Zariski fermé de codimension au moins c. On remarque que par la théorie de Hodge mixte (*cf.* [30]), la filtration de coniveau géométrique est une filtration par sous-structures de Hodge.

Soient X^{Zar} la variété munie de la topologie de Zariski et X^{an} la variété analytique correspondante (munie de la topologie analytique). On dispose de l'application 'identité' suivante, qui est continue :

$$f : X^{an} \to X^{Zar}.$$

On considère la suite spectrale de Leray associée :

$$E_2^{p,q} = H^p(X^{Zar},\mathscr{H}^q) \Rightarrow H^{p+q}(X^{an},\mathbf{Q}),$$

où $\mathscr{H}^q := R^qf_*\underline{\mathbf{Q}}_{X^{an}}$ est le faisceau Zariski associé au préfaisceau $U \mapsto H^q(U^{an},\mathbf{Q})$. Bloch et Ogus [17] démontrent que la filtration de Leray associée sur la cohomologie de Betti est exactement la filtration de coniveau géométrique définie ci-dessus.

Maintenant on introduit une autre filtration de coniveau sur la cohomologie, qui est fortement reliée à la précédente. Soit L une structure de Hodge (pure, effectif) de poids k. On définit son *coniveau de Hodge* comme le plus grand entier $c \in \mathbf{N}$, tel que $L^{k,0} = L^{k-1,1} = \cdots = L^{k-c+1,c-1} = 0$. On peut alors définir la *filtration de coniveau de Hodge* sur la cohomologie de Betti d'une variété projective lisse X :

$$N^c_{Hdg}H^k(X,\mathbf{Q}) := \sum_{\substack{L\subset H^k(X,\mathbf{Q}) \text{ sous-S.H.} \\ \mathrm{coniveau}(L)\geq c}} L.$$

Example 0.2.1. Une intersection complète lisse dans $\mathbf{P}^N$ de multi-degré $d_1 \leq \cdots \leq d_r$ a la cohomologie de degré moitié de coniveau de Hodge $\geq c$ si et seulement si $\sum_{i=1}^r d_i + (c-1)d_r \leq N$.

La relation suivante entre ces deux filtrations de coniveau est facile :

Lemme 0.2.2. *On a toujours une inclusion :* $N^cH^k(X,\mathbf{Q}) \subset N^c_{Hdg}H^k(X,\mathbf{Q})$.

Démonstration. Par la théorie de Hodge mixte,

$$\mathrm{Ker}\left(H^k(X) \xrightarrow{j^*} H^k(X\backslash Y)\right) = \mathrm{Im}\left(H_{2n-k}(Y)(-n) \xrightarrow{i_*} H^k(X)\right)$$

est égal à

$$\mathrm{Im}\left(H_{2n-k}(\widetilde{Y})(-n) \xrightarrow{\widetilde{i}_*} H^k(X)\right) = \mathrm{Im}\left(H^{k-2c}(\widetilde{Y})(-c) \xrightarrow{\widetilde{i}_*} H^k(X)\right),$$

où i, j sont des inclusions et $\tau : \widetilde{Y} \to Y$ est une désingularisation de Y, $\widetilde{i} = i\circ\tau$. L'inclusion dans l'énoncé découle du fait que $H^{k-2c}(\widetilde{Y},\mathbf{Q})(-c)$ est de coniveau $\geq c$. □

Grothendieck [36] propose la conjecture suivante :

Conjecture 0.2.3 (Conjecture de Hodge généralisée)**.** *La filtration de coniveau géométrique coïncide avec la filtration de coniveau de Hodge :*

$$N^c H^k(X, \mathbf{Q}) = N^c_{Hdg} H^k(X, \mathbf{Q}).$$

Plus concrètement, si une sous-structure de Hodge $L \subset H^k(X, \mathbf{Q})$ est de coniveau c, alors il existe un sous-ensemble Y Zariski fermé de codimension au moins c tel que L est supportée sur Y.

La conjecture de Hodge usuelle est le cas où $k = 2c$.

0.2.2 Une invitation au Chapitre 1

Mon premier résultat, qui est présenté dans Chapitre 1, est un essai vers la direction de la conjecture de Hodge généralisée 0.2.3 ci-dessus.

Soit X une variété projective lisse complexe de dimension n. Soit D un diviseur gros sur X. Alors par la relation bilinéaire de Hodge-Riemann, le noyau

$$L := \mathrm{Ker}\left(\cup[D] : H^{n-1}(X) \to H^{n+1}(X)\right)$$

est une sous-structure de coniveau de Hodge au moins 1 (voir l'introduction du Chapitre 2 pour l'argument). On se propose à vérifier la prédiction de la conjecture de Hodge généralisée suivante :
Question : Soit L la sous-structure de Hodge de $H^{n-1}(X)$ définie ci-dessus. Est-elle supportée sur un diviseur de X ?

La réponse est positive en admettant la conjecture standard de Lefschetz, qui est une conjecture beaucoup plus faible. Le résultat principal de Chapitre 1 est une généralisation pour une classes de cohomologie grosse :

Théorème 0.2.4 (=Theorem 1.0.2)**.** *Soit X une variété projective lisse de dimension n, $k \in \{0, 1, \cdots, n\}$ un entier. Soit $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ une classe de cohomologie grosse (i.e. la somme d'un cycle effectif et une intersection des diviseurs amples). Soit L le noyau de l'application 'cup-produit avec γ' :*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Alors si on suppose la conjecture standard de Lefschetz, L est supportée sur un diviseur de X : i.e.

$$L \subset \mathrm{Ker}\left(H^k(X, \mathbf{Q}) \to H^k(X \backslash Z, \mathbf{Q})\right)$$

pour un sous-ensemble algébrique fermé Z de codimension 1 dans X.

L'outil crucial est encore une fois la relation bilinéaire de Hodge-Riemann. Voir Chapitre 1 pour les détailles.

0.2.3 Décomposition de la diagonale et le théorème de Mumford

On explique maintenant la technique de *décomposition de la diagonale* suivante initiée par Bloch et Srinivas [18], dont la démonstration s'appuie sur la technique d'étaler l'équivalence rationnelle et la dénombrabilité du schéma d'Hilbert (*cf.* [68]).

Théorème 0.2.5 (Décomposition I)**.** *Soit X une variété projective lisse complexe de dimension n, soit $j : Y \hookrightarrow X$ un sous-ensemble algébrique fermé. Si les 0-cycles de X sont supportés sur Y, i.e. l'application $j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ est surjective, alors il existe $m \in \mathbf{N}_+$, et une décomposition dans $\mathrm{CH}_n(X \times X)$:*

$$m\Delta_X = Z_1 + Z_2,$$

où Z_1 est un cycle supporté sur $Y \times X$, et Z_2 est supporté sur $X \times D$ pour $D \subsetneq X$ un sous-ensemble fermé de codimension ≥ 1.

Ce théorème nous permet d'obtenir des résultats sur la structure de Hodge à partir d'un contrôle sur le groupe de Chow :

Théorème 0.2.6 (Mumford généralisé I). *Soit X une variété projective lisse complexe de dimension n. Si les 0-cycles de X sont supportés sur un sous-ensemble fermé de codimension $\geq c$. Alors $H^n(X), H^{n-1}(X), \cdots, H^{n-c+1}(X)$ sont de coniveau géométrique au moins 1. En particulier, ces structures de Hodge sont de coniveau de Hodge ≥ 1 :*

$$H^{n,0}(X) = H^{n-1,0}(X) = \cdots = H^{n-c+1,0}(X) = 0.$$

Démonstration. On voit la décomposition de la diagonale dans le théorème précédent comme une égalité des correspondances de X vers X. En faisant agir cette égalité sur la cohomologie, on déduit les annulations souhaitées. □

Voir §0.2.6 pour autres applications de la technique de décomposition de la diagonale et aussi des généralisations pour les petites diagonales.

La décomposition de Bloch-Srinivas ci-dessus a été généralisé par Paranjape [59], Laterveer [49] en itérant le même argument. Je présente un énoncé un peu plus fort que les leurs.

Théorème 0.2.7 (Décomposition II). *Soit X une variété projective lisse complexe de dimension n. Si on a des sous-ensemble algébrique fermés $Y_0, \cdots, Y_n$ de X tels que pour tout $0 \leq i \leq n$, $\mathrm{CH}_i(X)$ est supporté sur Y_i. Alors il existe $m \in \mathbf{N}_+$ et une décomposition dans $\mathrm{CH}_n(X \times X)$:*

$$m\Delta_X = Z_0 + Z_1 + \cdots + Z_n,$$

où Z_i est un n-cycle supporté sur $Y_i \times W_i$ avec W_i un sous-ensemble Zariski fermé de codimension i.

Démonstration. D'abord, par la décomposition de Bloch-Srinivas, l'hypothèse que $\mathrm{CH}_0(X)$ est supporté sur Y_0 nous fournit une décomposition de la forme :

$$m_0\Delta_X = Z_0 + Z_1' \quad \text{dans} \quad \mathrm{CH}_n(X \times X),$$

où Z_0 est un cycle supporté sur $Y_0 \times W_0$, $W_0 := X$, et Z_1' est un cycle supporté sur $X \times W_1$ pour un sous ensemble fermé W_1 satisfaisant $\mathrm{codim}_X(W_1) = 1$.
Quitte à prendre une désingularisation, on suppose pour simplicité que W_1 est lisse, et on note toujours par Z_1' le n-cycle sur $X \times W_1$ dont l'image dans $\mathrm{CH}_n(X \times X)$ est Z_1' ci-dessus. Donc Z_1' est une famille de 1-cycle de X paramétré par W_1. Par l'hypothèse que $\mathrm{CH}_1(X)$ est supporté sur Y_1, le théorème de Bloch-Srinivas implique qu'il existe $m_1 \in \mathbf{N}_+$, et une décomposition dans $\mathrm{CH}_n(X \times W_1)$ suivante :

$$m_1 Z_1' = Z_1 + Z_2',$$

où Z_1 est supporté sur $Y_1 \times W_1$ et Z_2' est supporté sur $X \times W_2$ avec W_2 un sous-ensemble fermé satisfaisant $\mathrm{codim}_{W_1}(W_2) \geq 1$. En itérant l'argument, et on arrive à une décomposition désirée (où $m = \prod m_i$). □

De la même manière, en regardant la décomposition comme une égalité des correspondances de X vers X, on obtient :

Théorème 0.2.8 (Mumford généralisé II). *Soit X une variété projective lisse complexe de dimension n. Si $\mathrm{CH}_i(X)$ est supporté sur un sous-ensemble Zariski fermé de dimension d_i, pour $0 \leq i \leq n$. Alors on a les annulations de groupes de Hodge suivantes :*

$$H^{p,q}(X) = 0, \quad \forall p > \max\{d_0, \cdots, d_q\}.$$

Démonstration. Étant donnée une classe de cohomologie $\alpha \in H^{p,q}(X)$ avec $p > \max\{d_0, \cdots, d_q\}$, on a par la décomposition du Théorème précédent :

$$m\alpha = \sum_{i=0}^{n} (Z_i)_*(\alpha).$$

Or si $i \leq p$, alors $\alpha|_{\widetilde{Y_i}} \in H^{p,q}(\widetilde{Y_i})$, où $\widetilde{Y_i}$ est une désingularisation de Y_i. Mais $H^{p,q}(Y_i) = 0$ parce que $\dim(\widetilde{Y_i}) = d_i < p$ par l'hypothèse. Donc $\alpha|_{\widetilde{Y_i}}$ ainsi $(Z_i)_*(\alpha)$ est nulle.

Si $i > p$, alors $(Z_i)_*(\alpha) \in \operatorname{Im}\left(H^{p+q-2i}(\widetilde{W_i}) \to H^{p+q}(X)\right)$, où $\widetilde{W_i}$ est une désingularisation de W_i. Mais l'image est évidement de coniveau $\geq i > p$, donc $\operatorname{Im}\left(H^{p+q-2i}(\widetilde{W_i}) \to H^{p+q}(X)\right) \cap H^{p,q}(X) = 0$. En conséquence, on a aussi $(Z_i)_*(\alpha) = 0$ dans ce cas. Du coup $m\alpha = 0$. Comme m est strictement positif, $\alpha = 0$. □

On note le cas particulier suivant où $d_i = i$ pour tout $0 \leq i \leq c-1$.

Corollaire 0.2.9. *Soit X une variété projective lisse complexe de dimension n avec* $\mathrm{CH}_i(X)_{\mathbf{Q},hom} = 0$ *pour* $0 \leq i \leq c-1$. *Alors* $H^k(X)^{\perp_{alg}} = 0$ *est de coniveau géométrique (donc aussi le coniveau de Hodge)* $\geq c$ *pour tout k, où* $H^k(X)^{\perp_{alg}}$ *est l'orthogonal des classes algébriques par rapport au cup-produit avec* $H^{2n-k}(X)$.

0.2.4 Conjecture de Bloch et ses généralisations

Les réciproques des théorèmes de Mumford généralisés sont appelées Conjectures de Bloch.

Conjecture 0.2.10 (Bloch généralisée)**.** *Soit X une variété projective lisse complexe de dimension n. Si* $H^k(X)^{\perp_{alg}} = 0$ *est de coniveau* $\geq c$ *pour tout k, alors* $\mathrm{CH}_i(X)_{\mathbf{Q},hom} = 0$ *pour* $0 \leq i \leq c-1$.

La version originale de la conjecture de Bloch est pour la surface :

Conjecture 0.2.11 (Bloch)**.** *Soit S une surface projective lisse complexe. Si* $H^{2,0}(S) = 0$, *alors* $\mathrm{CH}_0(X)_{hom} \to \mathrm{Alb}(X)$ *est un isomorphisme.*

Remarques 0.2.12. (1) La conjecture de Bloch pour les surfaces de dimension de Kodaira au plus 1 a été démontrée dans [16]. Pour les surfaces de type général, *cf.* [65], [64].
(2) La conjecture de Bloch généralisée et la conjecture de Hodge généralisée sont équivalentes pour les intersections complètes en supposant la conjecture standard, *cf.* [74].

Ce qui nous intéresse est la version *équivariante* :

Conjecture 0.2.13 (Bloch généralisée équivariante)**.** *Soit X une variété projective lisse complexe de dimension n munie d'un automorphisme f d'ordre fini. Si f agit comme l'identité sur* $H^{p,q}$ *pour tout* $p < c$ *ou* $q < c$ *avec* $p \neq q$. *Alors f agit aussi comme l'identité sur* $\mathrm{CH}_i(X)_{hom,\mathbf{Q}}$ *pour tout* $i < c$.

En fait, cette version équivariante pour être considérée comme un exemple de la généralisation de Conjecture 0.2.10 aux motifs de Chow suivante :

Conjecture 0.2.14 (Bloch généralisée motivique)**.** *Soit M un motif de Chow effectif sur* $\mathbf{C}$. *Si* $H^{p,q}(M) = 0$ *pour tout* $p < c$ *ou* $q < c$ *avec* $p \neq q$, *alors* $\mathrm{CH}_i(M)_{\mathbf{Q},hom} = 0$ *pour tout* $0 \leq i \leq c-1$.

En effet, dans la situation de Conjecture 0.2.13, si on note m l'ordre de f, alors par l'hypothèse, le motif de Chow $M := (X, \Delta_X - \frac{\Delta_X+\Gamma_f+\cdots+\Gamma_{f^{m-1}}}{m})$ a cohomologie de coniveau $\geq c$, et la conclusion est équivalente à $\mathrm{CH}_i(M)_{hom,\mathbf{Q}} = 0$ pour tout $i < c$.

On remarque que les conjectures de Bloch (usuelle, généralisée, équivariante, motivique) sont toutes des conséquences de la conjecture suivante due à Bloch et Beilinson (*cf.* [13], [68], [6], [45]) :

Conjecture 0.2.15 (Bloch-Beilinson)**.** *Pour toute variété complexe projective lisse, il existe une filtration sur le groupe de Chow à coefficient rationnel : $F^\bullet \mathrm{CH}^k(X)_\mathbf{Q}$, $0 \leq k \leq n := \dim X$, satisfaisant :*

(*i*) *$F^\bullet$ est stable par correspondances, en particulier, elle est fonctorielle en push-forwards, pull-backs.*

(*ii*) $F^0 \mathrm{CH}^k(X)_\mathbf{Q} = \mathrm{CH}^k(X)_\mathbf{Q}$ *;* $F^1 \mathrm{CH}^k(X)_\mathbf{Q} = \mathrm{CH}^k(X)_{\mathbf{Q},hom}$.

(*iii*) $F^i \mathrm{CH}^*(X)_\mathbf{Q} \bullet F^j \mathrm{CH}^*(X)_\mathbf{Q} \subset F^{i+j} \mathrm{CH}^*(X)_\mathbf{Q}$.

(*iv*) $\mathrm{Gr}^r_F \mathrm{CH}^k(X)_\mathbf{Q}$ *est* contrôlé *par la structure de Hodge $H^{2k-r}(X, \mathbf{Q})$. Plus précisément,* $\mathrm{Gr}^r_F \mathrm{CH}^k(X)_\mathbf{Q} = 0$ *si $H^{2k-r}(X)$ est de coniveau de Hodge $> k - r$.*

(*v*) $F^{k+1} \mathrm{CH}^k(X) = 0$.

0.2.5 Une invitation au Chapitre 3

Le troisième sujet de recherche de ma thèse, qui est le contenu de Chapitre 3, est dans le cadre des conjectures de Bloch généralisées discutées ci-dessus, surtout la version équivariante, à savoir, Conjecture 0.2.13.

Le point de départ est le théorème suivant dû à Huybrechts et Voisin, qui est impliqué par la conjecture de Bloch équivariante 0.2.13 en dimension 2.

Théorème 0.2.16 (Voisin [76], Huybrechts [39])**.** *Un automorphisme symplectique d'ordre fini d'une surface K3 projective agit trivialement sur son groupe de Chow des 0-cycles.*

Pour généraliser ce théorème en dimension supérieure, on introduit d'abord la généralisation des surfaces K3. Une variété projective lisse complexe est dite *hyperkählérienne* (ou *symplectique holomorphe irréductible*), si elle est simplement connexe et admet une 2-forme holomorphe symplectique (*i.e.* non-dégénérée en chaque point), unique à un scalaire près. Voir plus loin §0.3.3 pour une introduction plus détaillée sur ce type de variétés.

D'une façon analogue au théorème précédent, le but est d'étudier la conjecture suivante qui est une conséquence de Conjecture 0.2.13.

Conjecture 0.2.17. *Un automorphisme symplectique d'ordre fini d'une variété hyperkählérienne projective agit trivialement sur son groupe de Chow des 0-cycles.*

Le résultat principal de Chapitre 3 est la vérification de cette conjecture pour la famille des varétés hyperkählériennes projective construite par Beauville et Donagi [11] sous l'hypothèse que l'automorphisme est polarisé.

Théorème 0.2.18 (=Theorem 3.0.5)**.** *Soit $X \subset \mathbf{P}^5$ une hypersurface cubique lisse de dimension 4. Soit f un automorphisme* [1] *de X. Si l'action induite $\hat{f}$ sur la variété des droites $F(X)$, préserve la forme symplectique, alors $\hat{f}$ agit sur* $\mathrm{CH}_0(F(X))$ *comme l'identité.*
D'une manière équivalente, tout automorphisme symplectique polarisé de $F(X)$ agit trivialement sur $\mathrm{CH}_0(F(X))$*, où $F(X)$ est munie de la polarisation de Plücker.*

1. f est automatiquement d'ordre fini.

La technique de la démonstration est d'étaler les cycles algébriques en mettant les objets en famille, comme dans l'article de Voisin [74]. Voir Chapitre 3 pour les détailles.

On déduit du Théorème ci-dessus la conséquence suivante :

Corollaire 0.2.19 (= Corollary 3.0.7)**.** *Sous la même hypothèse du théorème précédent. Un automorphisme polarisé symplectique de la variété des droites $F(X)$ d'une hypersurface cubique X de dimension 4, agit trivialement sur* $\mathrm{CH}_2(F(X))_{\mathbf{Q},hom}$.

0.2.6 Décomposition de la petite diagonale et anneaux de Chow

Pour motiver, notons encore quelques applications (*cf.* [18]) de la technique de décomposition de la diagonale $\Delta_X \subset X \times X$ présentée dans §0.2.3, Théorème 0.2.5. Soit X une variété projective lisse complexe.

- La conjecture de Hodge pour $H^4(X, \mathbf{Q})$ est vraie, si $\mathrm{CH}_0(X)$ est supporté sur un sous-ensemble algébrique fermé de dimension 3.
- Si $\mathrm{CH}_0(X)$ est supporté sur un sous-ensemble algébrique fermé de dimension 1, alors $\mathrm{CH}^2(X)$ est représentable au sens de Mumford.
- $\mathrm{Griff}^2(X)$ et $\mathrm{Griff}_1(X)$ sont de torsion si $\mathrm{CH}_0(X)$ est supporté sur un sous-ensemble algébrique fermé de dimension 2, où Griff est le *groupe de Griffiths*, qui est par définition le groupe des cycles algébriques homologiquement triviaux modulo l'équivalence algébrique.

On résume ce qui précède par le principe suivant :
Principe 1 : Une décomposition de la diagonale $\Delta_X \subset X \times X$ mette en précision la mesure que le motif de X est dégénéré, qui nous permet de contrôler la taille de $\mathrm{CH}^*(X)$.

Comme une généralisation, on s'intéresse aussi à *la petite diagonale*

$$\delta_X := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

D'une manière parallèle, on a le principe suivant :
Principe 2 : Une décomposition de la petite diagonale $\delta_X \subset X \times X \times X$ nous permet de contrôler la structure multiplicative de l'anneau de Chow $\mathrm{CH}^*(X)$.

En effet, on peut voir une décomposition de la petite diagonale comme une égalité des correspondances de $X \times X$ vers X. Pour deux cycles algébriques $z, z' \in \mathrm{CH}^*(X)$, l'action de δ_X sur le produit extérieur $z \times z'$ nous donne exactement $z \bullet z' \in \mathrm{CH}^*(X)$, où $\bullet$ est le produit d'intersection pour l'anneau de Chow $\mathrm{CH}^*(X)$.

Le premier exemple dans cette direction est dû à Beauville et Voisin :

Théorème 0.2.20 ([12])**.** *Soit S une surface K3 projective, $c_S \in \mathrm{CH}_0(S)$ le 0-cycle représenté par un point sur une courbe rationnelle de S. Alors*
(1) On a la décomposition suivante dans $\mathrm{CH}_2(S \times S \times S)$,

$$\delta_S = \Delta_{12} + \Delta_{13} + \Delta_{23} - S \times c_S \times c_S - c_S \times S \times c_S - c_S \times c_S \times S$$

où Δ_{12} est la grande diagonale $\{(x, x, c_S) \mid x \in S\}$, et Δ_{13}, Δ_{23} sont définies par analogue.
(2) Le produit d'intersection de deux diviseurs est toujours proportionnel à c_S dans $\mathrm{CH}_0(S)$, i.e.

$$\mathrm{Im}\Big(\mathrm{Pic}(S) \times \mathrm{Pic}(S) \xrightarrow{\bullet} \mathrm{CH}_0(S)\Big) = \mathbf{Z} \cdot c_S.$$

On remarque d'abord que (2) est remarquable en comparant le théorème de Mumford qui implique que $\mathrm{CH}_0(S)$ est 'non-représentable' ou 'de dimension infinie' (*cf.* [54]). On rappèle aussi que par notre Principe 2 ci-dessus, (2) est une conséquence immédiate de (1). Néanmoins, (2) est plus facile à démontrer et utilisé pour établir (1) dans [12].

0.2.7 Une invitation au Chapitre 2

Le deuxième thème d'étude dans ma thèse, qui est le sujet de Chapitre 2, est de généraliser le résultat de Beauville-Voisin (Théorème 0.2.20) : d'une part une décomposition de la petite diagonale pour une variété à fibré canonique trivial et d'autre part l'implication sur la structure multiplicative de l'anneau de Chow : une dégénérescence du produit d'intersection.

Le premier résultat dans cette direction est dû à Voisin, qui concerne les hypersurfaces de type Calabi-Yau dans un espace projectif.

Théorème 0.2.21 ([75]). *Soit $X \subset \mathbf{P}^{n+1}$ une hypersurface générale de type Calabi-Yau. Soit $h := c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X)$ la classe de section d'hyperplan et $h_i := \mathrm{pr}_i^*(h) \in \mathrm{CH}^1(X^3)$ pour $i = 1, 2, 3$. On note aussi $c_X := \frac{h^n}{n+2} \in \mathrm{CH}_0(X)_{\mathbf{Q}}$. Alors*
(1) On a une décomposition de la petite diagonale dans $\mathrm{CH}_n(X^3)_{\mathbf{Q}}$:

$$\delta_X = \frac{1}{(n+2)!}\Gamma + \Delta_{12} + \Delta_{13} + \Delta_{23} + P(h_1, h_2, h_3),$$

où $\Delta_{12} = \Delta_X \times c_X$, et Δ_{13}, Δ_{23} sont définies par analogue ; P est un polynôme homogène de degré $2n$; et $\Gamma := \bigcup_{t\in F(X)} \mathbf{P}^1_t \times \mathbf{P}^1_t \times \mathbf{P}^1_t \subset X^3$, où $F(X)$ est la variété des droites de X, et $\mathbf{P}^1_t$ est la droite correspondante à $t \in F(X)$.
(2) Le produit d'intersection des deux cycles algébriques de dimensions complémentaires et strictement positives est toujours proportionnel à c_X dans $\mathrm{CH}_0(X)_{\mathbf{Q}}$, i.e. pour tout $i, j \in \mathbf{N}^*$ avec $i + j = n$, on a

$$\mathrm{Im}\left(\mathrm{CH}^i(X)_{\mathbf{Q}} \times \mathrm{CH}^j(X)_{\mathbf{Q}} \xrightarrow{\bullet} \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q}\cdot c_X.$$

Les résultats principaux du Chapitre 2 sont deux généralisations du théorème précédent. Le premier est pour les intersections complètes de type Calabi-Yau dans un espace projectif.

Théorème 0.2.22 (*cf.* Theorem 2.0.7)**.** *Soit X une intersection complète générale de type Calabi-Yau de multi-degré $d_1 \geq \cdots \geq d_r$, dans un espace projectif. Alors*

(1) Dans $\mathrm{CH}_n(X^3)$, on a une décomposition de la petite diagonale :

$$\left(\prod_{i=1}^{r}(d_i!)\right)\cdot \delta_X = \Gamma + j_{12*}(Q(h_1,h_2)) + j_{13*}(Q(h_1,h_2)) + j_{23*}(Q(h_1,h_2)) + P(h_1,h_2,h_3),$$

où Q et P sont des polynômes homogènes à coefficients entiers ; Γ est défini comme dans le théorème précédent ; $h_i \in \mathrm{CH}^1(X^3)$ ou $\mathrm{CH}^1(X^2)$ est le pull-back de $h = c_1(\mathscr{O}_X(1))$ par la i-ème projection ; et j_{12}, j_{13}, j_{23} sont les inclusions des grandes diagonales $X^2 \hookrightarrow X^3$.

(2) Pour tout $k, l \in \mathbf{N}^*$ avec $k + l = n$,

$$\mathrm{Im}\left(\bullet : \mathrm{CH}^k(X)_{\mathbf{Q}} \times \mathrm{CH}^l(X)_{\mathbf{Q}} \to \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q}\cdot h^n,$$

où $h = c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X)$.

Dans ce théorème, comme Théorème 0.2.21, (2) est déduit de (1) par la méthode résumé dans notre Principe 2 ci-dessus. Pour établir la décomposition, on prend la stratégie de [75]. Les intersections d'excès posent certaines difficultés supplémentaires. Voir Chapitre 2 pour les détailles.

La deuxième généralisation concerne à nouveau une hypersurface dans un espace projectif, mais *la plus petite diagonale* dans une puissance plus grande de cette hypersurface. Voir la première partie du Théorème 2.0.8 pour une telle décomposition. On présente ici seulement sa conséquence sur la structure multiplicative de l'anneau de Chow :

Théorème 0.2.23 (*cf.* Theorem 2.0.8)**.** *Soit X une hypersurface lisse dans $\mathbf{P}^{n+1}$ de degré $d \geq n+2$. On note $k := d+1-n \geq 3$. Alors pour $i_1, i_2, \cdots, i_{k-1} \in \mathbf{N}^*$ avec $\sum_{j=1}^{k-1} i_j = n$, l'image*

$$\operatorname{Im}\left(\mathrm{CH}^{i_1}(X)_{\mathbf{Q}} \times \mathrm{CH}^{i_2}(X)_{\mathbf{Q}} \times \cdots \times \mathrm{CH}^{i_{k-1}}(X)_{\mathbf{Q}} \xrightarrow{\bullet} \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n.$$

On remarque que ceci est une conséquence de la conjecture de Bloch-Beilinson 0.2.15 mentionnée avant.

0.3 Variétés à fibré canonique trivial

Dans cette dernière section de l'introduction, on résume la théorie de base d'une classe de variétés qui consiste l'interêt principal de la thèse, à savoir, les variétés à fibré canonique trivial.

0.3.1 Théorème de décomposition de Beauville

Pour étudier les variétés à fibré canonique trivial, le point de départ est le théorème de Yau suivant :

Théorème 0.3.1 ([77])**.** *Soit X une variété complexe compacte de type kählérien* [2]*. Si sa première classe de Chern (topologique) est nulle : $c_1(T_X) = 0$, alors il existe une unique métrique kählérienne ω dans chaque classe de kähler, dont la courbure de Ricci correspondante est nulle :* $\mathrm{Ric}(X, \omega) = 0$.

Remarque 0.3.2. Pour une variété kählérienne compacte, la courbure de Ricci n'est autre que la courbure, à un scalaire non-nul près, du fibré canonique muni de la métrique induite. De manière équivalente, la représentation d'holonomie est restreinte dans le groupe unitaire spécial $\mathrm{SU}(n)$, où n est la dimension de X.

On a ainsi le corollaire suivant :

Corollaire 0.3.3. *Soit X une variété compacte de type kählérien, les trois conditions suivantes sont équivalentes :*
(1) La première classe de Chern (topologique) est nulle : $c_1(T_X) = 0$;
(2) Le fibré canonique K_X est trivial ;
(3) Il existe une métrique kählérienne Ricci-plate.

Remarque 0.3.4. Par le théorème de Bogomolov-Tian-Todorov, l'espace de Kuranishi Def_X des déformations d'une telle variété X est lisse de dimension $h^{1,n-1}(X)$.

En utilisant la classification de groupe d'holonomie, Beauville [9] décompose une telle variété, à un revêtement étale fini près, en un produit de facteurs *irréductibles* :

Théorème 0.3.5 (Beauville [9])**.** *Soit X une variété complexe compacte de type kählérien, dont la première classe de Chern est nulle : $c_1(T_X) = 0$. Alors il existe un revêtement étale fini $\widetilde{X}$ de X isomorphe à un produit :*

$$\widetilde{X} \simeq T \times \prod Y_i \times \prod Z_j,$$

où T est un tore complexe, Y_i sont des variétés de Calabi-Yau simplement connexe, et Z_j sont des variétés hyperkählériennes compactes.

2. On dit qu'une variété complexe compacte est *de type kählérien*, si elle admet une métrique kählérienne.

Une esquisse de la démonstration : Grâce au théorème de Yau, on peut supposer que X est une variété kählérienne Ricci-plate, de dimension n, donc la représentation d'holonomie est contenue dans le sous groupe $SU(n)$. Par le théorème de de Rham, son revêtement universel se décompose (isométriquement) :

$$\hat{X} \simeq \mathbf{C}^k \times \prod X_l,$$

avec chaque X_l compacte [3], Ricci plate, simplement connexe, ayant la représentation d'holonomie irréductible et contenue dans $SU(d_l)$, $d_l = \dim X_l$. D'après la classification de groupe d'holonomie [14], le groupe d'holonomie de chaque X_l est soit égal à $SU(d_l)$, soit égal à $Sp(d_l/2)$. Par les interprétations holonomiques des variétés de Calabi-Yau et hyperkählériennes (voir plus bas), on a :

$$\hat{X} \simeq \mathbf{C}^k \times \prod Y_i \times \prod Z_j$$

avec Y_i et Z_j comme dans le théorème. En utilisant le fait que les groupes des automorphismes de Y_i et Z_j sont tous discrets, on peut arriver à un revêtement étale fini satisfaisant la conclusion. □

Avant de décrire les trois types de variété mentionnés, on note qu'une surface K3 est à la fois Calabi-Yau et hyperkählérienne, et une courbe elliptique est à la fois un tore complexe et de Calabi-Yau, donc pour obtenir l'unicité de la décomposition dans le théorème et faciliter la présentation suivante, on va insister qu'*une variété de Calabi-Yau est de dimension au moins 3*.

On commence à préciser les trois types de variété. D'abord,

Tores complexes : un *tore complexe* $T = V/\Lambda$ est par définition le quotient d'un espace vectoriel complexe V par un réseau Λ.

0.3.2 Variétés de Calabi-Yau

Par définition, une variété kählérienne compacte Y de dimension $n \geq 3$ est de type *Calabi-Yau*, si le fibré canonique K_Y est trivial et $H^0(Y, \Omega_Y^i) = 0$ pour tout $0 < i < n$.

Voici une autre manière de les caractériser, qui utilise le principe de Bochner : sur une variété kählérienne compacte Ricci-plate, tout champ de tenseurs holomorphe est parallèle.

Lemme 0.3.6 (Interprétation holonomique de la condition de Calabi-Yau)**.** *Soit Y une variété complexe compacte de type kählérien de dimension $n \geq 3$. Alors Y est une variété de Calabi-Yau si et seulement si elle admet une métrique kählérienne dont le groupe d'holonomie est égal à* $SU(n)$.

Exemple 0.3.7. On donne quelques exemples de variétés de Calabi-Yau :
(1) Intersection complète lisse de hypersurfaces de degré $(d_1, d_2, \cdots, d_r)$ dans $\mathbf{P}^m$, avec $\sum_{i=1}^r d_i = m + 1$.
(2) Plus généralement, soit X une variété de Fano, alors un membre général Y dans le système linéaire $|-K_X|$ est une variété de Calabi-Yau.

0.3.3 Variétés hyperkählériennes compactes

Par définition, une variété kählérienne compacte X de dimension $2n$ est *hyperkählérienne* ou *symplectique holomorphe irréductible*, si X est simplement connexe, et $H^0(X, \Omega_X^2)$ est de dimension 1, engendré par une 2-forme holomorphe ϕ, qui est non-dégénérée en chaque point. La 2-forme holomorphe ϕ est appelée la forme *symplectique* de X, qui est définie à un scalaire près.

Une définition équivalente :

3. La compacité est assurée par le théorème de Cheeger-Gromoll [21].

Lemme 0.3.8 (Interprétation holonomique de la condition hyperkählérienne)**.** *Soit X une variété complexe compacte de type kählérien de dimension $2n$. Alors X est hyperkählérienne si et seulement si elle admet une métrique kählérienne dont le groupe d'holonomie est égal à* $\mathrm{Sp}(n)$.

Par le principe de Bochner et le principe d'holonomie, on déduit que $H^0(X, \Omega_X^{2i}) = \mathbf{C} \cdot \phi^i$, pour $0 \leq i \leq n$, où ϕ est la 2-forme holomorphe symplectique de X.

Exemple 0.3.9. Donnons quelques exemples de variété hyperkählérienne :
(1) Les surfaces K3.
(2) Soit S une surface K3, alors le schéma de Hilbert ponctuel $S^{[r]}$ est une variété hyperkählérienne, *cf.* [9].
(3) Soit A un tore complexe de dimension 2, alors la fibre du morphisme somme $s : A^{[r]} \to A$, notée $K_r := s^{-1}(0)$ est une variété hyperkählérienne, *cf.* [9].
(4) Comme une déformation d'une variété hyperkählérienne reste hyperkählérienne, on obtient des familles de variété hyperkählérienne en déformant les exemples précédents.
(5) O'Grady construit une famille de variété hyperkählérienne de dimension 10, comme une désingularisation d'une compactification lisse de l'espace de module des fibrés vectoriels stables de rang 2 avec classes de Chern $c_1 = 0, c_2 = 4$ sur une surface K3, *cf.* [56].
(6) De façon analogue, dans [57], O'Grady construit une famille de dimension 6 à partir de l'espace de module des fibrés vectoriels stables de rang 2 avec classes de Chern $c_1 = 0, c_2 = 2$ sur un tore complexe de dimension 2, et prenant une fibre pour la rendre simplement connexe.

Forme quadratique de Beauville-Bogomolov

Un outil important pour étudier les variétés hyperkählériennes est la forme quadratique de Beauville-Bogomolov. Soit X une variété hyperkählérienne de dimension $2n$, avec la 2-forme symplectique holomorphe ϕ, on définit une forme quadratique q_X sur $H^2(X, \mathbf{C})$ par la formule suivante :

$$q_X(\alpha) := \frac{n}{2} \int_X \alpha^2 (\phi\overline{\phi})^{n-1} + (1-n) \int_X \alpha\phi^{n-1}\overline{\phi}^n \int_X \alpha\phi^n\overline{\phi}^{n-1}$$

D'une manière équivalente, si on décompose α par type : $\alpha = a\phi + \omega + b\overline{\phi}$ où $a, b \in \mathbf{C}$ et ω est de type (1,1), alors

$$q_X(\alpha) = ab + \frac{n}{2} \int_X \omega^2 (\phi\overline{\phi})^{n-1}.$$

Beauville [9] montre que q_X est non-dégénérée et à un scalaire réel positif près, elle provient d'une forme quadratique entière primitive sur $H^2(X, \mathbf{Z})$ de signature $(3, b_2 - 3)$.

Théorème de Torelli local

D'après le théorème de Bogomolov-Tian-Todorov, l'espace de Kuranishi Def_X est lisse de dimension $h^{1,1}(X)$. On note $\mathscr{X} \to \mathrm{Def}_X$ pour la famille universelle, et pour chaque $t \in \mathrm{Def}_X$, on note X_t la déformation correspondante avec ϕ_t la forme symplectique sur X_t. On peut définir *l'application de période* :

$$\mathscr{P} : \mathrm{Def}_X \to \mathbf{P}H^2(X, \mathbf{C})$$

qui associe $t \in \mathrm{Def}_X$ à la classe $[u_t^*(\phi_t)]$ dans $\mathbf{P}H^2(X, \mathbf{C})$, où $u : X \times \mathrm{Def}_X \xrightarrow{\simeq} \mathscr{X}$ est une trivialization difféomorphique locale. On remarque que l'application de période est holomorphe.

Théorème 0.3.10 (Torelli local [9])**.** *L'image de l'application de période $\mathscr{P}$ est dans Ω, qui est un ouvert d'un quadratique défini par :*

$$\Omega = \left\{\alpha \in \mathbf{P}H^2(X, \mathbf{C}) | q_X(\alpha) = 0, q_X(\alpha, \overline{\alpha}) > 0\right\}. \tag{1}$$

De plus, l'application $\mathscr{P} : \mathrm{Def}_X \to \Omega$ *est étale* [4].

Démonstration. L'application tangente de $\mathscr{P} : \mathrm{Def}_X \to \mathbf{P}\,H^2(X, \mathbf{C})$ est

$$T_0\,\mathrm{Def}_X = H^1(X, T_X) \xrightarrow{\iota} \mathrm{Hom}_{\mathbf{C}}(H^{2,0}(X), H^{1,1}(X) \oplus H^{0,2}(X)),$$

donnée par le produit intérieur. Par la transversalité de Griffiths et la définition de Ω, l'application tangente de $\mathscr{P} : \mathrm{Def}_X \to \Omega$ est

$$H^1(X, T_X) \xrightarrow{\iota} \mathrm{Hom}_{\mathbf{C}}(H^{2,0}(X), H^{1,1}(X)), \tag{2}$$

encore donnée par le produit intérieur. Puisque la forme symplectique ϕ induit un isomorphisme entre le fibré tangent et le fibré cotangent : $\iota_\phi : T_X \xrightarrow{\simeq} \Omega^1_X$, ainsi un isomorphisme $\iota_\phi : H^1(X, T_X) \to H^1(X, \Omega^1_X)$, d'où, (2) est bien un isomorphisme, *i.e.* $\mathscr{P} : \mathrm{Def}_X \to \Omega$ est localement isomorphe. □

L'analogue du théorème de Torelli global de surfaces K3 ne vaut plus pour les variétés hyperkählériennes de dimension supérieure ([26]). Par contre, une version plus faible du théorème de Torelli global est établie récemment, *cf.* [40].

Déformations polarisées

À partir d'une variété hyperkählérienne *projective* X, grâce au théorème de Torelli local (Théorème 0.3.10), on peut identifier (locallement) Def_X, par l'application de période $\mathscr{P}$, avec Ω defini par (1). Un problème naturel est caractériser les déformations projectives.

Fixons une polarisation L de X, *i.e.* un fibré en droite ample, considérons

$$\Omega_L := \{\phi \in \Omega | q_X(\phi, c_1(L)) = 0\}\,, \tag{3}$$

qui est une hypersurface dans Ω. Comme une classe entière $a \in H^2(X_t, \mathbf{Z})$ est de type $(1, 1)$ si et seulement si elle satisfait $q_{X_t}(a, \phi_t) = 0$, donc les déformations X_t représentées par $t \in \Omega_L$ sont les deformations de X pour lesquelles $L_t := u_t^{-1*}(L)$ reste de type (1,1). En particulier, X_t est projective, polarisée par L_t, pour tout $t \in \Omega_L$. On appelle cette famille Ω_L la *déformation de X respectant la polarisation L*. On remarque que deux familles de déformations respectant L_1 et L_2 sont transverse dans Def_X si et seulement si $c_1(L_1)$ et $c_1(L_2)$ sont linéairement indépendantes dans $H^2(X, \mathbf{C})$.

Corollaire 0.3.11. *Les déformations projectives sont denses dans* Def_X.

Exemple 0.3.12. Voici quelques familles *complètes* [5] de déformations polarisées de $S^{[2]}$ pour une surface K3 projective S projective.
(1) La variété des droites contenues dans une hypersurface cubique de dimension 4, *cf.* [11].
(2) La variété des sommes de puissances d'une hypersurface cubique de dimension 4, *cf.* [43], [44].
(3) Un revêtement double d'une certaine hypersurface sextique de dimension 4 (*EPW-sextique*), *cf.* [58].
(4) Soit V un espace vectoriel de dimension 10, on fixe une 3-forme σ sur V générale, la sous variété de la Grassmannienne $\mathrm{Gr}(6, V)$ constituée des sous-espaces de dimension 6 sur lesquels la restriction de σ est nulle est hyperkählérienne de dimension 4, *cf.* [28].
Récemment, une nouvelle famille complète de déformations projectives des variétés hyperkählériennes de dimension 8 est trouvée, *cf.* [50].

4. *i.e.* localement isomorphe, néamoins, il faut faire attention que l'espace de déformation n'est pas forcément séparé.
5. C'est-à-dire, la famille est une hypersurface dans l'espace de Kuranishi de la forme précédente, *i.e.* une famille de déformation maximale respectant une certaine polarisation.

Chapter 1

On the coniveau of certain sub-Hodge structures

Résumé Par les relations bilinéaires de Hodge-Riemann, le noyau du cup produit avec une classe de cohomologie grosse est une sous-structure de Hodge de coniveau de Hodge au moins 1. La conjecture de Hodge généralisée de Grothendieck prédit que ce noyau est supporté sur un diviseur. On démontre cette prédiction en admettant la conjecture standard de Lefschetz.

Abstract We study the generalized Hodge conjecture for certain sub-Hodge structure defined as the kernel of the cup product map with a big cohomology class, which is of Hodge coniveau at least 1. As predicted by the generalized Hodge conjecture, we prove that the kernel is supported on a divisor, assuming the Lefschetz standard conjecture.

1.0 Introduction

Given a smooth projective variety X defined over $\mathbf{C}$, let $H^k(X, \mathbf{Q})$ be its k-th Betti cohomology group, which carries a pure Hodge structure of weight k. We can ask the philosophical question: how much information about the geometry of the variety, for example a knowledge of its subvarieties, can be extracted from the shape of certain associated transcendental objects, namely the Hodge structures on its cohomology groups? The generalized Hodge conjecture formulates a precise such relationship.

Recall that the *Hodge coniveau* of a weight k (pure) Hodge structure $(L, L^{p,q})$ is defined to be the largest integer $c \leq \lfloor \frac{k}{2} \rfloor$ such that $L^{0,k} = L^{1,k-1} = \cdots = L^{c-1,k-c+1} = 0$. If for any integer c, we define $N^c_{\mathrm{Hdg}} H^k(X, \mathbf{Q})$ as the sum of all the sub-Hodge structures in $H^k(X, \mathbf{Q})$ of Hodge coniveau at least c, we obtain the *Hodge coniveau filtration* on $H^k(X, \mathbf{Q})$. On the other hand, in terms of the topology of algebraic subvarieties of X, we also have the so-called *arithmetic filtration* or *coniveau filtration* $N^c H^k(X, \mathbf{Q})$ (*cf.* [35] [36] [17]), where $N^c H^k(X, \mathbf{Q})$ consists of the cohomology classes supported on some algebraic subset of codimension at least c, here *supported* on a closed subset means the class becomes zero when it is restricted to the open complement. The following inclusion (*cf.* §1.1) gives a first relation between the two filtrations:

$$N^c H^k(X, \mathbf{Q}) \subset N^c_{\mathrm{Hdg}} H^k(X, \mathbf{Q}). \tag{1.1}$$

In his famous paper [36], Grothendieck conjectures that the two filtrations in fact coincide, more precisely:

Conjecture 1.0.1 (Grothendieck amended generalized Hodge conjecture)**.** *Let X be a smooth projective variety of dimension n, $0 \leq k \leq n$ be an integer, and $L \subset H^k(X, \mathbf{Q})$ be a sub-Hodge structure of Hodge coniveau at least c, then there exists a closed algebraic subset Z of codimension at least c, such that*

$$L \subset \operatorname{Ker}\left(j^* : H^k(X, \mathbf{Q}) \to H^k(X \backslash Z, \mathbf{Q})\right),$$

where $j : X\backslash Z \to X$ is the natural inclusion.

Note that the usual Hodge conjecture is the case $k = 2c$.
The usual Hodge conjecture already has many theoretical consequences. For example, it implies that a morphism of Hodge structures between the cohomology groups of two smooth projective varieties is always induced by an algebraic correspondence (*cf.* Remark 1.1.4). In particular, it implies the Lefschetz standard conjecture (*cf.* §1.3.1), which says that the inverse of the hard Lefschetz isomorphism

$$L_X^i : H^{n-i}(X, \mathbf{Q}) \xrightarrow{\simeq} H^{n+i}(X, \mathbf{Q})$$

is induced by an algebraic cycle in $\mathrm{CH}^{n-i}(X \times X)_{\mathbf{Q}}$. The generalized Hodge conjecture has strong implications about the Chow groups, let us just mention [65] [72] [74].

The usual Hodge conjecture is widely open. The known cases of it include $k = 2c = 0, 2, 2n-2, 2n$ (thus for varieties of dimension at most 3), varieties with cellular decomposition (Grassmannians, flag varieties, or more generally, quotients of reductive linear algebraic groups by parabolic subgroups), cubic 4-folds ([78] [11]) *etc.* While for the generalized Hodge conjecture, besides the aforementioned cases, very few are known. One class of known cases concerns algebraic varieties with an automorphism group, see for example [8] and [65]. As far as we know, besides these and some results about abelian varieties (*cf.* [1] [2] [3]), there are no general results verifying the conjecture for a proper sub-Hodge structure.

In this chapter, we try to understand such a sub-Hodge structure situation. Our starting point is a discovery of a sub-Hodge structure of Hodge coniveau ≥ 1, which we describe here in the case of divisors for simplicity.

For an ample divisor A on an n-dimensional smooth projective variety X, the hard Lefschetz isomorphism tells us in particular that $\operatorname{Ker}(\cup[A] : H^{n-1}(X) \to H^{n+1}(X))$ vanishes. Now if we weaken the positivity assumption, namely consider a *big* divisor $D = A + E$, where A is an ample divisor, and $E = \sum_i m_i E_i$ is an effective divisor, then in general,

$$L := \operatorname{Ker}\left(\cup[D] : H^{n-1}(X) \to H^{n+1}(X)\right) \tag{1.2}$$

could be non-trivial, for instance: (see also Example 1.2.2)

Example. Let $X = \mathrm{Bl}_y Y \xrightarrow{\tau} Y$ be the blow-up of a point in a smooth projective 3-fold Y, and $D := \tau^*(\mathscr{O}_Y(1))$ be the pull back of an ample divisor on Y. Then D is big, while $L = \operatorname{Ker}(\cup[D] : H^2(X) \to H^4(X))$ is generated by the fundamental class of the exceptional divisor $[E] \in H^2(X)$.

Although L does not vanish in general, we still expect the positivity condition on D implies some control on L. In the above example we can readily see that L is supported on a divisor, thus of Hodge coniveau ≥ 1 in particular. Following the idea of [72], we get in general:

Observation (Lemma 1.2.3): If D is big, then L defined as in (1.2) is of Hodge coniveau at least 1.

Indeed, for any class $\alpha \in H^{n-1,0}(X)$ (in particular it is primitive), if $D \cup \alpha = 0$ but $\alpha \neq 0$, then $0 = \int_X D\alpha\overline{\alpha} = \int_X [A]\alpha\overline{\alpha} + \int_X [E]\alpha\overline{\alpha} = \int_X [A]\alpha\overline{\alpha} + \sum_i m_i \int_{\widetilde{E_i}} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)}$, where $\tau_i : \widetilde{E_i} \to E_i$

is a resolution of singularities for each i. However the second Hodge-Riemann bilinear relation (*cf.* [67]) gives

$$(-1)^{\frac{n(n-1)}{2}} i^{n-1} \int_X [A]\alpha\overline{\alpha} > 0$$

and for any i

$$(-1)^{\frac{n(n-1)}{2}} i^{n-1} \int_{\widetilde{E}_i} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)} \geq 0$$

Summing up these inequalities, we have a contradiction, therefore $\alpha = 0$, thus proving the observation.

Regarding the generalized Hodge conjecture, we ask the natural

Question (Conjecture 1.2.4). Can we prove that the kernel L of cup product with a big class is supported on a divisor of X?

We answer this question in this chapter assuming the Lefschetz standard conjecture. Here is the main theorem, where the role of big divisor classes is played by the more general notion of *big cohomology classes* (*cf.* Definition 1.2.1):

Theorem 1.0.2 (=Theorem 1.3.11)**.** *Let X be a smooth projective variety of dimension n, $k \in \{0, 1, \cdots, n\}$ be an integer, and $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ be a big cohomology class. Let L be the kernel of the following morphism of 'cup product with γ':*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Then assuming the Lefschetz standard conjecture, L is supported on a divisor of X, that is,

$$L \subset \mathrm{Ker}\left(H^k(X, \mathbf{Q}) \to H^k(X \backslash Z, \mathbf{Q})\right)$$

for some closed algebraic subset Z of codimension 1.

The proof consists of three steps:

- Proposition 1.2.5 realizes $L(1)$ effectively as a sub-Hodge structure of the degree $(k-2)$ cohomology of some other smooth projective variety. This step reduces the question to the usual Hodge conjecture;
- We use the standard conjecture to construct adjoint correspondences (§1.3.2) to get a divisor-supported sub-Hodge structure which is transverse to the orthogonal complement of L as in (1.10);
- We use the adjoint correspondences (§1.3.2) to construct the orthogonal projector onto L in Proposition 1.3.12.

Here is the structure of this chapter. In §2, besides some general remarks on the generalized Hodge conjecture, we give a description of the gap between the usual and the generalized Hodge conjectures (Lemma 1.1.3). In §3 we introduce the coniveau 1 sub-Hodge structure mentioned above, which is our main object of study, and we show that the generalized Hodge conjecture 1.0.1 is satisfied for it assuming the usual Hodge conjecture. In §4 we begin by making some general remarks concerning the Lefschetz standard conjecture, then we give the basic construction of the so-called *adjoint correspondences*, and finally we prove our main theorem 1.0.2. In the last section §5, we discuss some unconditional results and give a reinterpretation of our main result in the language of motivated cycles of Y. André.

1.1 Generalities of the Generalized Hodge Conjecture

In this section, we introduce the generalized Hodge conjecture and make a comparison with the usual Hodge conjecture.

First of all, let us recall some standard terminologies [1]. Let m be an integer.

- The *Tate Hodge structure* $\mathbf{Q}(m)$ is the pure Hodge structure of weight $-2m$, with the underlying rational vector space $\mathbf{Q}$, and with the Hodge decomposition concentrated at bidegree $(-m,-m)$.
- The *Tate twist* $L(m)$ of a pure Hodge structure L of weight k is defined to be the tensor product $L\otimes\mathbf{Q}(m)$, which is a Hodge structure of weight $k-2m$. More concretely, the underlying rational vector space is L, while the Hodge decomposition $L(m)\otimes_{\mathbf{Q}}\mathbf{C}=\oplus_{p+q=k-2m}L(m)^{p,q}$ is given by $L(m)^{p,q}=L^{p+m,q+m}$.
- A weight k pure Hodge structure $(L,L_{\mathbf{C}}=\oplus_{p+q=k}L^{p,q})$ is called *effective*, if $L^{p,q}=0$ when $p<0$ or $q<0$.

Here is the important notion of *Hodge coniveau* of a Hodge structure.

Definition 1.1.1 (Hodge coniveau)**.** Let

$$(L,L_{\mathbf{C}}=\bigoplus_{\substack{p+q=k\\ p,q\geq 0}}L^{p,q})$$

be an effective pure Hodge structure of weight k. The *Hodge coniveau* of L is defined to be the largest integer c such that the Tate twist $L(c)$ is an effective pure Hodge structure of weight $k-2c$. In other words, the Hodge decomposition takes the following form

$$L_{\mathbf{C}}=L^{c,k-c}\oplus L^{c+1,k-c-1}\oplus\cdots\oplus L^{k-c,c}$$

with $L^{c,k-c}\neq 0$.

Note that the Hodge coniveau of a non-zero effective pure Hodge structure of weight k is always $\leq\lfloor\frac{k}{2}\rfloor$.

Given a smooth projective variety X of dimension n, and a closed algebraic subset Z of codimension $\geq c$, it is an easy consequence of the strictness of morphisms between mixed Hodge structures (*cf.* [30]) that

$$\mathrm{Ker}\left(H^k(X)\xrightarrow{j^*}H^k(X\backslash Z)\right)=\mathrm{Im}\left(H_{2n-k}(Z)(-n)\xrightarrow{i_*}H^k(X)\right)$$

is equal to

$$\mathrm{Im}\left(H_{2n-k}(\widetilde{Z})(-n)\xrightarrow{\widetilde{i}_*}H^k(X)\right)=\mathrm{Im}\left(H^{k-2c}(\widetilde{Z})(-c)\xrightarrow{\widetilde{i}_*}H^k(X)\right),$$

where i,j are the inclusions, $\tau:\widetilde{Z}\to Z$ is a resolution of singularities of Z, $\widetilde{i}=i\circ\tau$, and all the (co-)homology groups are with rational coefficients. Since $H^{k-2c}(\widetilde{Z},\mathbf{Q})(-c)$ is a Hodge structure of Hodge coniveau $\geq c$, we deduce that $\mathrm{Im}\left(\widetilde{i}_*\right)$ hence $\mathrm{Ker}(j^*)$ is a sub-Hodge structure of Hodge coniveau at least c.

1. We will ignore the usual factor $2\pi i$, which makes the formulations in algebraic de Rham cohomology and in Betti cohomology compatible. But we will not make any such comparison argument in this chapter.

This explains in particular the inclusion (1.1) in the introduction:

$$N^c H^k(X, \mathbf{Q}) \subset N^c_{\mathrm{Hdg}} H^k(X, \mathbf{Q}),$$

while Grothendieck's generalized Hodge conjecture 1.0.1 states the reverse inclusion. In the situation of the conjecture, we say that L is *supported on* Z.

Remarks 1.1.2. The generalized Hodge conjecture is widely open.

- The cases $k = 0, 1$ are trivial, and the case of $k = 2$ follows from the Lefschetz theorem on (1,1)-classes.

- For $k \leq n$, if we view the cohomology group $H^{2n-k}(X, \mathbf{Q})$ as of weight k via the twist $\mathbf{Q}(n-k)$, the analogous conjecture for $H^{2n-k}(X, \mathbf{Q})$ follows from the conjecture for $H^k(X, \mathbf{Q})$ by hard Lefschetz isomorphisms.

- Note that the usual Hodge conjecture is exactly the case when $k = 2i$ and $c = i$, since to give a sub-Hodge structure of Hodge coniveau i in $H^{2i}(X, \mathbf{Q})$ amounts to give a Hodge class of degree $2i$ up to a constant scalar. The known cases of the usual Hodge conjecture include $k = 2c = 0, 2, 2n-2, 2n$ (thus for varieties of dimension at most 3), varieties with cellular decomposition (Grassmannians, flag varieties), cubic 4-folds ([78] [11]) and so on.

- For general complete intersections in projective spaces, the generalized Hodge conjecture for the middle cohomology is equivalent to the generalized Bloch conjecture, assuming the Lefschetz standard conjecture (*cf.* [74]).

- [65] deals with some complete intersection surfaces with an automorphism group. See also [8] for a similar result about Calabi-Yau 3-folds.

- There are some results for abelian varieties (*cf.* [1] [2] [3]).

We would like to make the following well-known remark which says that the gap between the usual Hodge conjecture and the generalized Hodge conjecture is the problem of looking for an *effective realization* of the Tate twist of the sub-Hodge structure. For more general remarks to the generalized Hodge conjecture, we refer to the papers [62] [60].

Lemma 1.1.3 (Hodge conjecture *vs.* Generalized Hodge conjecture)**.** *Let X be a smooth projective variety of dimension n, and $L \subset H^k(X, \mathbf{Q})$ be a sub-Hodge structure of Hodge coniveau at least c. Assume the following condition:*
$(*)$ *There exists a smooth projective variety Y, such that $L(c)$ is a sub-Hodge structure of $H^{k-2c}(Y, \mathbf{Q})$. Then the usual Hodge conjecture for $Y \times X$ implies the generalized Hodge conjecture for L.*

Before the proof of the lemma, let us recall the following fundamental interpretation of a morphism between two Hodge structures as a Hodge class in their Hom-space viewed as a Hodge structure (*cf.* [67]):

Remark 1.1.4. Let $k_1, k_2 \in \mathbf{Z}$ be of the same parity, and we set $c = \frac{k_2 - k_1}{2} \in \mathbf{Z}$. Let L_1, L_2 be two rational pure Hodge structures of weights k_1, k_2 respectively. The canonical identification $\mathrm{Hom}_{\mathbf{Q}}(L_1, L_2) = L_1^* \otimes_{\mathbf{Q}} L_2$ induces on $\mathrm{Hom}_{\mathbf{Q}}(L_1, L_2)$ a Hodge structure of weight $k_2 - k_1$. Then a linear map $f \in \mathrm{Hom}_{\mathbf{Q}}(L_1, L_2)$ is a morphism of Hodge structures of bidegree (c, c) if and only if f is a Hodge class of degree $2c$ with respect to this natural Hodge structure. In the geometric setting, let X, Y be smooth projective varieties of dimension n, m respectively, and $f : H^{k_1}(X, \mathbf{Q}) \to H^{k_2}(Y, \mathbf{Q})$ be a $\mathbf{Q}$-linear map, then f is a morphism of Hodge structures of bidegree (c, c) if and only if f is a Hodge class of degree $2c$ in $\mathrm{Hom}_{\mathbf{Q}}\left(H^{k_1}(X, \mathbf{Q}), H^{k_2}(Y, \mathbf{Q})\right) = H^{k_1}(X, \mathbf{Q})^* \otimes_{\mathbf{Q}} H^{k_2}(Y, \mathbf{Q}) \cong$

$H^{2n-k_1}(X,\mathbf{Q})(n)\otimes_{\mathbf{Q}} H^{k_2}(Y,\mathbf{Q})$, which is a direct factor of $H^{2n-k_1+k_2}(X\times Y,\mathbf{Q})(n)$ by the Künneth formula. For such Hodge class f, if moreover there is an algebraic cycle $\mathscr{Z}\in \mathrm{CH}^{n+c}(X\times Y)_{\mathbf{Q}}$ such that the fundamental class $[\mathscr{Z}]\in H^{2n+2c}(X\times Y,\mathbf{Q})$ coincides with f when projecting to the Künneth factor $H^{2n-k_1}(X,\mathbf{Q})(n)\otimes_{\mathbf{Q}} H^{k_2}(Y,\mathbf{Q})$, then we say that f is *algebraic*, meaning that f is induced by an algebraic correspondence. In particular, the (usual) Hodge conjecture implies that any morphism of Hodge structures $f: H^{k_1}(X,\mathbf{Q})\to H^{k_2}(Y,\mathbf{Q})$ is in fact algebraic.

Proof of Lemma 1.1.3. (*cf.* [62].) Since the Hodge structure $H^{k-2c}(Y,\mathbf{Q})$ is polarizable, $L(c)$ is a direct factor of $H^{k-2c}(Y,\mathbf{Q})$ in the category of Hodge structures. In particular, there is a projection $H^{k-2c}(Y,\mathbf{Q})\twoheadrightarrow L(c)$, which is a morphism of Hodge structures. Twisting it by $\mathbf{Q}(-c)$, and composing with the inclusion of L into $H^k(X,\mathbf{Q})$, we get a morphism of Hodge structures

$$H^{k-2c}(Y,\mathbf{Q})(-c)\to H^k(X,\mathbf{Q})$$

with image L. Now apply the usual Hodge conjecture for $Y\times X$ (*cf.* Remark 1.1.4), we conclude that this morphism of Hodge structures is algebraic, *i.e.* it is the correspondence induced by an algebraic cycle $\mathscr{Z}\in \mathrm{CH}_{n-c}(Y\times X)$. Therefore, $L=\mathrm{Im}\left([\mathscr{Z}]_*: H^{k-2c}(Y)(-c)\to H^k(X)\right)$ is supported on $Z:=\mathrm{Supp}\,(\mathrm{pr}_2(\mathscr{Z}))$, the support of the image of $\mathscr{Z}$ under the projection to X. Clearly, every irreducible component of Z is of dimension at most $\dim(\mathscr{Z})=n-c$, hence of codimension at least c. □

Remarks 1.1.5. The condition ($*$) in the above lemma is always satisfied when $k=2c$ (trivial) or $k=2c+1$ (thanks to the anti-equivalence of categories between weight 1 effective rational Hodge structures and abelian varieties up to isogenies). Moreover, by the Lefschetz theorem of hyperplane sections, we can reduce to the case of $\dim(Y)=k-2c$ by taking successive general hyperplane sections on Y.

1.2 Kernel of the Cup Product Map with Big Classes

For a smooth projective variety X, let $H^{2i}(X,\mathbf{Q})_{\mathrm{alg}}$ be the $\mathbf{Q}$-subspace of $H^{2i}(X,\mathbf{Q})$ generated by the fundamental classes of algebraic cycles of codimension i. In $H^{2i}(X,\mathbf{Q})_{\mathrm{alg}}$ sits the *effective cone* generated by the effective algebraic cycles of codimension i. Making an analogue of the divisor case, we define a cohomology class to be *big*, if it is in the interior (when passing to the real coefficients) of the effective cone. Here is the practical definition that we will use in this chapter.

Definition 1.2.1 (Big cohomology classes)**.** Let X be a smooth projective variety, and let $0\le i\le \dim(X)$ be an integer. A cohomology class $\gamma\in H^{2i}(X,\mathbf{Q})(i)$ is called *big*, if some of its positive multiples is of the form

$$m\gamma=[A]^i+[E]\ \text{ in } H^{2i}(X,\mathbf{Z})(i),\quad m\in\mathbf{N}^*,$$

where A is an ample divisor, E is an effective algebraic cycle of codimension i, and [-] means the cohomology class of an algebraic cycle.

To simplify the notation, we will mostly suppress the Tate twists from now on, except when we want to highlight it.

Note that if the class $\gamma\in H^{2i}(X,\mathbf{Q})$ is 'ample' in the sense that $\gamma\in[A]^i\cdot\mathbf{Q}_{>0}$ for some ample divisor A, then the hard Lefschetz theorem says $\cup\gamma: H^{n-i}(X,\mathbf{Q})\to H^{n+i}(X,\mathbf{Q})$ is an isomorphism; in particular, the kernel is trivial. But when γ is only big, the kernel could be non-trivial as the following example shows.

Example 1.2.2. Let V be a smooth projective variety of dimension n with a smooth subvariety Z of codimension $c \geq 2$. Let $X := \mathrm{Bl}_Z V \xrightarrow{\tau} V$ be the blow-up of V along Z, and E be the exceptional divisor:

$$\begin{array}{ccc} E & \xrightarrow{\iota} & \mathrm{Bl}_Z V \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle \tau} \\ Z & \xrightarrow[i]{} & V \end{array}$$

We consider $\gamma = \tau^*(A)$, the pull-back of an ample divisor class A on V. Thanks to the following formula for the cohomology of blow-ups (*cf.* [67] Theorem 7.31):

$$\tau^* \oplus \bigoplus_{i=0}^{c-2} \iota_* \xi^i p^* : H^{n-1}(V) \oplus \bigoplus_{i=0}^{c-2} H^{n-3-2i}(Z) \xrightarrow{\simeq} H^{n-1}(X)$$

where $\xi = \mathscr{O}_E(1)$, we find that

$$\mathrm{Ker}\left(\cup\gamma : H^{n-1}(X) \to H^{n+1}(X)\right) \simeq \bigoplus_{i=0}^{c-2} \mathrm{Ker}\left(\cup A|_Z : H^{n-3-2i}(Z) \to H^{n-1-2i}(Z)\right),$$

which does not vanish in general.

Despite $\mathrm{Ker}(\cup\gamma) \neq 0$ in general, we still expect the positivity assumption on γ would imply the kernel is 'small' in certain sense. For instance in the above example, we observe that the kernel is in fact supported in the exceptional divisor E; in particular, $\mathrm{Ker}(\cup\gamma)$ is of Hodge coniveau at least 1.

The following Lemma 2.1.21 generalizes this example. This observation is the starting point of the chapter. The idea of using the Hodge-Riemann bilinear relations goes back to [72].

Lemma 1.2.3 (Observation)**.** *Let X be a smooth projective variety of dimension n, $0 \leq k \leq n$ be an integer, and $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ be a big cohomology class. Let L be the kernel of the following morphism of 'cup product with γ':*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Then L is a sub-Hodge structure of $H^k(X, \mathbf{Q})$ of Hodge coniveau at least 1.

Proof. Replacing γ by a multiple if necessary, we can suppose that

$$\gamma = [A]^{n-k} + [E] \ \text{ in } H^{2n-2k}(X, \mathbf{Z}),$$

where $A = \mathscr{O}_X(1)$ is a general hyperplane section, and E is an effective algebraic cycle of codimension $n-k$. Since $\cup\gamma$ is clearly a morphism of Hodge structures, its kernel L is of course a sub-Hodge structure. Therefore to prove the Hodge coniveau 1 assertion, which means $L^{k,0} = 0$, it suffices to show that for any class $\alpha \in H^{k,0}(X)$, if $\gamma \cup \alpha = 0$, then $\alpha = 0$. Let α be such a class.

Let $E = \sum_i m_i E_i$ with $m_i \in \mathbf{N}^*$ be the decomposition into linear combination of prime divisors, and $\tau_i : \widetilde{E_i} \to E_i$ be a resolution of singularities for each i. As $\gamma \cup \alpha = 0$, we have

$$\begin{aligned} 0 &= \int_X \gamma\alpha\overline{\alpha} \\ &= \int_X [A]^{n-k}\alpha\overline{\alpha} + \int_X [E]\alpha\overline{\alpha} \\ &= \int_X [A]^{n-k}\alpha\overline{\alpha} + \sum_i m_i \int_{\widetilde{E_i}} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)} \end{aligned}$$

However, since α is primitive in $H^k(X, \mathbf{C})$, by the second Hodge-Riemann bilinear relation (*cf.* [67]),

$$(-1)^{\frac{k(k-1)}{2}} i^k \int_X [A]^{n-k} \alpha\overline{\alpha} \geq 0, \tag{1.3}$$

with equality holds only when $\alpha = 0$.

Similarly, since $\tau_i^*(\alpha)$ is also of type $(k, 0)$, in particular primitive in $H^k(\widetilde{E_i}, \mathbf{C})$, we have again by the second Hodge-Riemann bilinear relation that for each i,

$$(-1)^{\frac{k(k-1)}{2}} i^k \int_{\widetilde{E_i}} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)} \geq 0. \tag{1.4}$$

As the sum of the left hand sides of (1.3) and (1.4) is zero, we have an equality in (1.3), *i.e.* $\alpha = 0$, and hence L is of Hodge coniveau at least 1. □

Combining the above observation 1.2.3 with the generalized Hodge conjecture 1.0.1, one gets the following conjecture which is the main subject of the chapter.

Conjecture 1.2.4. *Let X be a smooth projective variety of dimension n, $0 \leq k \leq n$ be an integer, and $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ be a big cohomology class (in the sense of Definition 1.2.1). Let L be the kernel of the following morphism of 'cup product with γ':*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Then L is supported on a divisor of X, i.e. $L \subset \operatorname{Ker}\left(H^k(X, \mathbf{Q}) \to H^k(X\backslash Z, \mathbf{Q})\right)$ for some Z closed algebraic subset of codimension 1.

In the presence of Lemma 1.2.3, the cases of $k = 0, 1, n$ are trivial, and the case of $k = 2$ follows from the Lefschetz theorem on (1,1)-classes.

We would like to show first (see Corollary 1.2.6) that Conjecture 1.2.4 is implied by the usual Hodge conjecture. The key point is the following Proposition 1.2.5 of effective realization of $L(1)$. The argument appeared in C. Voisin's paper [72]. We reproduce her argument here since the construction will be useful in §1.3, where we will show that Conjecture 1.2.4 is in fact a consequence of the Lefschetz standard conjecture.

Proposition 1.2.5 (Effective realization)**.** *Let X, k, γ, L be as above. Then there exists a (not necessarily connected) smooth projective variety Y of dimension $k-1$ with a morphism $\mu : Y \to X$, such that the composition $L \hookrightarrow H^k(X, \mathbf{Q}) \xrightarrow{\mu^*} H^k(Y, \mathbf{Q})$ is injective.*
In particular, $L(1)$ is a sub-Hodge structure of $H^{k-2}(Y, \mathbf{Q})$.

Proof. Adopting the notations in Lemma 1.2.3, up to replacing γ by a positive multiple, we can assume

$$\gamma = [A]^{n-k} + [E] \text{ in } H^{2n-2k}(X, \mathbf{Z}),$$

where $A = \mathscr{O}_X(1)$ is a general hyperplane section and $E = \sum_i m_i E_i$ with $m_i \in \mathbf{N}^*$ is an effective algebraic cycle of dimension k, and $\tau_i : \widetilde{E_i} \to E_i$ be a resolution of singularities for each i. Let B be the intersection of $(n - k + 1)$ general hyperplane sections of X, and H_i be a general section of a very ample line bundle on $\widetilde{E_i}$, in particular, B and H_i are irreducible smooth projective varieties of dimension $k - 1$.

Let $Y := B \sqcup \bigsqcup_i H_i$ be their disjoint union, and $\mu : Y \to X$ be the natural morphism. We claim:
$(**)$ The composition $L \hookrightarrow H^k(X, \mathbf{Q}) \xrightarrow{\mu^*} H^k(Y, \mathbf{Q})$ is injective, *i.e.* $L \cap \operatorname{Ker}(\mu^*) = \{0\}$.

Indeed, since $L \cap \operatorname{Ker}(\mu^*)$ is a sub-Hodge structure, it suffices to show, for each (p, q) with $p + q = k$, that if $\alpha \in H^{p,q}(X)$ satisfies $\alpha \cup \gamma = 0$ and $\mu^*(\alpha) = 0$, then we have $\alpha = 0$. Suppose the contrary: $\alpha \neq 0$.

Since the composition $H^k(X,\mathbf{Q}) \xrightarrow{i_B^*} H^k(B,\mathbf{Q}) \xrightarrow{i_{B*}} H^{2n-k+2}(X,\mathbf{Q})$ is exactly the Lefschetz operator $[B] = [A]^{n-k+1}$ and the second morphism is an isomorphism by Lefschetz's hyperplane theorem, we find that $\mathrm{Ker}\left(i_B^* : H^k(X,\mathbf{Q}) \to H^k(B,\mathbf{Q})\right) = H^k(X,\mathbf{Q})_{\mathrm{prim}}$, where $i_B = \mu|_B$ is the natural inclusion of B into X; in particular, α is a primitive class of type (p,q).

As in the proof of Lemma 2.1.21, firstly we have

$$\begin{aligned} 0 &= \int_X \gamma\alpha\overline{\alpha} \qquad (\text{since } \gamma\cup\alpha = 0)\\ &= \int_X [A]^{n-k}\alpha\overline{\alpha} + \int_X [E]\alpha\overline{\alpha}\\ &= \int_X [A]^{n-k}\alpha\overline{\alpha} + \sum_i m_i \int_{\widetilde{E}_i} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)} \end{aligned}$$

However, since $\alpha \neq 0$ is primitive of type (p,q), by the second Hodge-Riemann bilinear relation (*cf.* [67]), we have

$$(-1)^{\frac{k(k-1)}{2}} i^{p-q} \int_X [A]^{n-k}\alpha\overline{\alpha} > 0.$$

Therefore, since the sum is zero, there exists i, such that

$$(-1)^{\frac{k(k-1)}{2}} i^{p-q} \int_{\widetilde{E}_i} \tau_i^*(\alpha)\overline{\tau_i^*(\alpha)} < 0.$$

By the second Hodge-Riemann bilinear relation of $\widetilde{E}_i$, we deduce that $\tau_i^*(\alpha)$ is NOT primitive in $H^k(\widetilde{E}_i,\mathbf{Q})$, *i.e.* $[H_i]\cup\tau_i^*(\alpha) \neq 0$. In particular, $(\mu|_{H_i})^*(\alpha) \neq 0$, giving a contradiction to the assumption that $\alpha \in \mathrm{Ker}(\mu^*)$. So the claim $(**)$ follows, and this is exactly what we want.

As for the last assertion, composing the injective morphism of Hodge structures obtained above $L \hookrightarrow H^k(Y,\mathbf{Q})$, with the inverse of the hard Lefschetz isomorphism (as Hodge structures) $H^{k-2}(Y,\mathbf{Q})(-1) \xrightarrow{\simeq} H^k(Y,\mathbf{Q})$, we get an inclusion of Hodge structures $L(1) \hookrightarrow H^{k-2}(Y,\mathbf{Q})$ as desired. □

Corollary 1.2.6. *Conjecture 1.2.4 is implied by the usual Hodge conjecture.*

Proof. To reach the generalized Hodge conjecture from the usual Hodge conjecture, we use Lemma 1.1.3 which explains the gap between them, so we only have to check in our situation the condition $(*)$ in Lemma 1.1.3.

However, the above Proposition 1.2.5 provides an inclusion of Hodge structures $L(1) \hookrightarrow H^{k-2}(Y,\mathbf{Q})$, and this is exactly the condition $(*)$ in Lemma 1.1.3. □

The rest of the chapter is devoted to the proof of the main theorem 1.0.2, which says that Conjecture 1.2.4 is in fact implied by an *a priori* much weaker conjecture, namely the Lefschetz standard conjecture.

1.3 Lefschetz Standard Conjecture implies Conjecture 1.2.4

We first recall the Lefschetz standard conjecture. Then in the second subsection we deal with the construction and the formal properties of the *adjoint* of an algebraic correspondence, which incorporates the strength of the Lefschetz standard conjecture; while in the third subsection, by rather formal arguments, we will deduce our main theorem 1.3.11 from Proposition 1.2.5, which embeds the Tate twist of the sub-Hodge structure in question into the cohomology of some smooth projective variety.

1.3.1 The Lefschetz Standard Conjecture

Here we gather some well-known general remarks concerning the Lefschetz standard conjecture, for a more complete treatment, see [47] [48]. Let X be a smooth projective variety of dimension n, $\mathscr{O}_X(1)$ be a very ample divisor which is chosen to be the polarization of X. Let $\xi = c_1(\mathscr{O}_X(1)) \in H^2(X, \mathbf{Q})$. Define the Lefschetz operator

$$L_X = \cup\xi : H^k(X, \mathbf{Q}) \to H^{k+2}(X, \mathbf{Q})$$

to be cup product with the first Chern class of the polarization. The hard Lefschetz theorem asserts that for any integer $k \in \{0, \cdots, n\}$, the morphism

$$L_X^{n-k} : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q})$$

is an isomorphism. Note that this isomorphism is in fact *algebraic* (see Remark 1.1.4), which means that it is the correspondence induced by a dimension k algebraic cycle $\Delta_{X*}(\mathscr{O}_X(1)^{n-k}) \in \mathrm{CH}_k(X \times X)$, where $\Delta_X : X \hookrightarrow X \times X$ is the diagonal inclusion. In his paper [37], Grothendieck conjectures that the inverse of the Lefschetz isomorphism is also algebraic.

Conjecture 1.3.1 (Lefschetz standard conjecture)**.** *In the above situation, there exists a codimension k algebraic cycle with rational coefficients $\mathscr{Z} \in \mathrm{CH}^k(X \times X)_{\mathbf{Q}}$, such that the induced correspondence*

$$[\mathscr{Z}]_* : H^{2n-k}(X, \mathbf{Q}) \to H^k(X, \mathbf{Q})$$

is the inverse of the isomorphism L_X^{n-k} defined above.

Remarks 1.3.2. We list some basic facts about the standard conjecture. Some of them will be used in the sequel.

- There are several equivalent versions of the Lefschetz standard conjecture (*cf.* [47] [48]). Besides the one stated above, let us just mention another equivalent one which says that the projectors $\pi_{L_X^i H^{k-2i}(X)_{\mathrm{prim}}}$, with respect to the Lefschetz decomposition $H^k(X) = \bigoplus_{i\geq \max\{0,k-n\}} L_X^i H^{k-2i}(X)_{\mathrm{prim}}$, are algebraic.

- The Lefschetz standard conjecture implies the Künneth standard conjecture which says that all the projectors $\pi^k : H^*(X) \twoheadrightarrow H^k(X) \hookrightarrow H^*(X)$ are algebraic.

- The Lefschetz standard conjecture is implied by the usual Hodge conjecture. Indeed, $(L_X^{n-k})^{-1}$ is a morphism of Hodge structures, by Remark 1.1.4 the corresponding cohomology class of $X \times X$ is a Hodge class (it is an *absolute* Hodge class [2] in fact), and the Hodge conjecture claims the existence of an algebraic cycle inducing $(L_X^{n-k})^{-1}$.

- The Lefschetz standard conjecture in degree 1, namely the algebraicity of $(L^{n-1})^{-1} : H^{2n-1}(X, \mathbf{Q}) \to H^1(X, \mathbf{Q})$, is implied by the Lefschetz theorem of (1,1)-class on $X \times X$. Thus the Lefschetz standard conjecture is known for curves and surfaces. Besides, other known cases include abelian varieties, generalized flag varieties. Note that this conjecture is stable by taking products, hyperplane sections (*cf.* [48]). Let us also mention the recent work [20] verifying this conjecture for certain type of irreducible holomorphic symplectic varieties.

2. Roughly speaking, they 'descend' with the field of definition of X, *cf.* [31].

1.3.2 Adjoint correspondences

For any smooth projective variety X of dimension n, with polarization $\mathscr{O}_X(1)$ and corresponding Lefschetz operator L_X, let us consider the following operator s_X on $H^*(X,\mathbf{Q})$, which changes the signs of the factors in the Lefschetz decomposition to retain the positivity property as in the primitive part.

Definition 1.3.3 (Operator s_X on $H^*(X)$)**.** For any integer $k \in \{0, \cdots, n\}$, the action of the operator s_X on $H^k(X) = \bigoplus_{0\leq i\leq \lfloor\frac{k}{2}\rfloor} L_X^i H^{k-2i}(X)_{\text{prim}}$ is defined as multiply by $(-1)^{\frac{k(k-1)}{2}} \cdot (-1)^i$ on the direct factor $L_X^i H^{k-2i}(X)_{\text{prim}}$ in the Lefschetz decomposition. Let the action of s_X on $H^{2n-k}(X)$ be the action induced from the one on $H^k(X)$ via the hard Lefschetz isomorphism.

Remarks 1.3.4. From the above definition, we note that

- s_X is an involution: $s_X \circ s_X = \text{id}$;
- s_X commutes with the Lefschetz operator $L_X \circ s_X = s_X \circ L_X$;
- The transpose of s_X is s_X;
- s_X is *rational*, *i.e.* it comes from a $\mathbf{Q}$-linear operator on $H^*(X,\mathbf{Q})$, the reason is that the Lefschetz decomposition is rational.

Lemma 1.3.5 (Algebraicity of s_X)**.** *Assuming the Lefschetz standard conjecture, the operator s_X is algebraic, i.e. it is induced by an algebraic cycle in* $\mathrm{CH}^n(X\times X)_{\mathbf{Q}}$.

Proof. We can write

$$s_X = \sum_{k=0}^{n}\Big(\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i \pi_{L_X^i H^{k-2i}(X)_{\text{prim}}}\Big)\cdot(-1)^{\frac{k(k-1)}{2}}\pi^k + \sum_{k=0}^{n}\Big(\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i \pi_{L_X^{n-k+i} H^{k-2i}(X)_{\text{prim}}}\Big)\cdot(-1)^{\frac{k(k-1)}{2}}\pi^{2n-k}.$$

By the first two points of Remark 1.3.2, assuming the Lefschetz standard conjecture, all the projectors appearing in the above formula, hence s_X itself, are algebraic. □

Usually, for $0 \leq k \leq n$, people use the pairing $< -,- >$ on $H^k(X)$ defined by

$$< x,y > := \int_X L_X^{n-k} xy \quad \text{for any } x,y \in H^k(X)$$

Although it is non-degenerate thanks to the hard Lefschetz isomorphism, it does not have the positivity property enjoyed by the primitive part any more. In the language of Hodge theory, we say that this pairing is NOT a polarization. To retain the positivity property, we define the following modified bilinear pairing on $H^k(X)$:

$$(x,y)_{H^k(X)} := \int_X L_X^{n-k} x \cdot s_X(y) \tag{1.5}$$

for any $x,y \in H^k(X)$. We sometimes suppress the subscript to write $(-,-)$ if we don't want to mention the Hodge structure explicitly. Then

- $(-,-)_{H^k(X)}$ is rational;
- $(x,y)_{H^k(X)} = (-1)^k (y,x)_{H^k(X)}$ for any $x,y \in H^k(X)$;

Moreover, by the Hodge-Riemann bilinear relations (*cf.* [67]), we find that for any $0 \leq k \leq n$, any $x \in H^{p,q}(X)$ and $y \in H^{p',q'}(X)$ with $p+q = p'+q' = k$, we have

- $(x,y) = \int_X L_X^{n-k}x \cdot s_X(y) = 0$ unless $(p,q) = (q',p')$;
- $(i^{p-q}x, \overline{x}) = \int_X L_X^{n-k} \cdot i^{p-q}x \cdot s_X(\overline{x}) > 0$ for any $0 \neq x \in H^{p,q}(X)$.

Therefore the bilinear pairing (1.5) on $H^k(X)$ is a *polarization* (*cf.* [67]) of the Hodge structure $H^k(X,\mathbf{Q})$. As for the Hodge structure $H^{2n-k}(X,\mathbf{Q})$, we use the polarization induced from the one on $H^k(X,\mathbf{Q})$ via the hard Lefschetz isomorphism.

Throughout this chapter, we will always use this polarization (1.5) on the cohomology groups of any polarized smooth projective variety.

Remark 1.3.6. The advantage of using the polarizations $(-,-)$ instead of the usual pairings $< -,- >$ can be summarized in the following very vague analogue: *as long as we stay*[3] *in the category of polarizable Hodge structures equipped with the polarizations above, to do linear algebra we can pretend that the spaces are euclidean spaces equipped with positive definite scalar products.* To illustrate this intuition as well as for later use, we want to recall here several basic properties of polarizations of Hodge structures, and more analogues can be found in the rest of this chapter. Let H be a Hodge structure with polarization $(-,-)$, and L be a sub-Hodge structure, then (*cf.* [67])

- $(-,-)|_L$ gives a polarization of L;
- $L^\perp$ is a sub-Hodge structure with polarization $(-,-)|_{L^\perp}$;
- $L \cap L^\perp = \{0\}$, thus $L \oplus L^\perp = H$.

Here comes the basic terminology that we will use in the following.

Proposition-Definition 1.3.7 (Adjoint correspondence)**.** *Let X, Y be smooth projective varieties of dimension n, m respectively, and $-n \leq r \leq m$ be an integer. Given $\mathscr{Z} \in \mathrm{CH}^{n+r}(X \times Y)_{\mathbf{Q}}$ an algebraic cycle with rational coefficients, viewed as a correspondence (cf. [34]) from X to Y, it induces morphisms on cohomology groups for any $k \in \{0,\cdots,2n\}$:*

$$C := [\mathscr{Z}]_* : H^k(X,\mathbf{Q}) \to H^{k+2r}(Y,\mathbf{Q}).$$

Assuming the Lefschetz standard conjecture, then there exists an algebraic cycle with rational coefficients $\mathscr{Z}^\dagger \in \mathrm{CH}^{m-r}(Y \times X)_{\mathbf{Q}}$, such that as a correspondence from Y to X, for any $k \in \{0,\cdots,2n\}$, the induced morphism on cohomology groups:

$$C^\dagger := [\mathscr{Z}^\dagger]_* : H^{k+2r}(Y,\mathbf{Q}) \to H^k(X,\mathbf{Q})$$

satisfies

$$(C\alpha,\beta)_{H^{k+2r}(Y)} = (\alpha, C^\dagger\beta)_{H^k(X)} \tag{1.6}$$

for any $\alpha \in H^k(X)$ and any $\beta \in H^{k+2r}(Y)$, where $(-,-)$ denotes the polarization of Hodge structures fixed in (1.5).
We call $\mathscr{Z}^\dagger$ an adjoint correspondence *of $\mathscr{Z}$, and also $C^\dagger$ the* adjoint (cohomological) correspondence *of C.*

3. That is, all the vector spaces considered are Hodge structures and all the relevant morphisms between them are morphisms of Hodge structures. In particular, all the subspaces should be sub-Hodge structures.

Proof. Since the Lefschetz standard conjecture implies the Künneth standard conjecture (*cf.* Remark 1.3.2), it suffices to construct for each $k \in \{0, \cdots, 2n\}$, an algebraic cycle $\mathscr{Z}_k^{\dagger} \in \mathrm{CH}^{m-r}(Y \times X)_{\mathbf{Q}}$ such that (1.6) is satisfied. Indeed, we could take $\mathscr{Z}^{\dagger} = \sum_k \pi_X^k \circ \mathscr{Z}_k^{\dagger} \circ \pi_Y^{k+2r}$, where $\circ$ means composition of correspondences (*cf.* [34]) and π are Künneth projectors which are algebraic by assumption.

Now we construct $\mathscr{Z}_k^{\dagger} \in \mathrm{CH}^{m-r}(Y \times X)_{\mathbf{Q}}$. For simplicity, we give the formula in the case that $0 \leq k \leq n$ and $0 \leq l := k + 2r \leq m$; the other cases follow immediately since we know that the inverse of the hard Lefschetz isomorphism is given by an algebraic correspondence. For any $\alpha \in H^k(X), \beta \in H^l(Y)$, we have:

$$\begin{aligned}
(C\alpha, \beta)_{H^l(Y)} &= \int_Y [\mathscr{Z}]_*(\alpha) \cdot L_Y^{m-l} \circ s_Y(\beta) && \text{(Definition (1.5))}\\
&= \int_X \alpha \cdot [\mathscr{Z}]^* \circ L_Y^{m-l} \circ s_Y(\beta) && \text{(projection formula)}\\
&= \int_X L_X^{n-k}\alpha \cdot s_X \circ (L_X^{n-k})^{-1} \circ s_X \circ [\mathscr{Z}]^* \circ L_Y^{m-l} \circ s_Y(\beta) && \text{(Remark 1.3.4)}\\
&= (\alpha, (L_X^{n-k})^{-1} \circ s_X \circ [\mathscr{Z}]^* \circ L_Y^{m-l} \circ s_Y(\beta))_{H^k(X)} && \text{(Definition (1.5))}
\end{aligned}$$

The s-operators and the inverse of the Lefschetz operator are supposed to be algebraic by the Lefschetz standard conjecture as the preceding lemma shows. We use the same notation to denote the algebraic cycles inducing them. Therefore, we can take

$$\mathscr{Z}^{\dagger} = (L_X^{n-k})^{-1} \circ s_X \circ {}^t\mathscr{Z} \circ L_Y^{m-l} \circ s_Y \tag{1.7}$$

where ${}^t\mathscr{Z} \in \mathrm{CH}^{n+r}(Y \times X)_{\mathbf{Q}}$ is the *transpose* of the correspondence $\mathscr{Z} \in \mathrm{CH}^{n+r}(X \times Y)_{\mathbf{Q}}$ (*cf.* [34]), and $\circ$ means the composition of correspondences. $C^{\dagger}$ is defined to be the cohomological correspondence induced by $\mathscr{Z}^{\dagger}$. □

Remark 1.3.8. Although the adjoint correspondence $\mathscr{Z}^{\dagger}$ of $\mathscr{Z}$ is not uniquely determined as an algebraic cycle modulo rational equivalence, the adjoint (cohomological) correspondence $C^{\dagger}$ of C is uniquely determined as a cohomological class in $H^*(Y \times X)$, since the polarization is non-degenerate.

As expected, we have immediately:

Lemma 1.3.9. *Let X, Y, r, $\mathscr{Z}$, C be as above, then*

- *For any $k \in \{0, \cdots, 2n\}$, $\alpha \in H^k(X)$, $\beta \in H^{k+2r}(Y)$, we have*

$$(C^{\dagger}\beta, \alpha)_{H^k(X)} = (\beta, C\alpha)_{H^{k+2r}(Y)}.$$

- *The operator $\dagger$ is an* involution*:*

$$C^{\dagger\dagger} = C.$$

- *If we have a third smooth projective variety Z, and an algebraic correspondence from Y to Z: $\mathscr{Z}' \in \mathrm{CH}(Y \times Z)_{\mathbf{Q}}$, and let C' be the corresponding cohomological correspondence, then we have a* functoriality*:*

$$(C'C)^{\dagger} = C^{\dagger}C'^{\dagger}$$

Proof. Indeed,

$$\begin{aligned}
(C^{\dagger}\beta, \alpha)_{H^k(X)} &= (-1)^k(\alpha, C^{\dagger}\beta)_{H^k(X)}\\
&= (-1)^{k+2r}(C\alpha, \beta)_{H^{k+2r}(Y)} \quad \text{(by (1.6))}\\
&= (\beta, C\alpha)_{H^{k+2r}(Y)}
\end{aligned}$$

gives the first assertion, and

$$(\beta, C^{\dagger\dagger}\alpha)_{H^{k+2r}(Y)} = (C^{\dagger}\beta, \alpha)_{H^k(X)} = (\beta, C\alpha)_{H^{k+2r}(Y)}$$

yields the second one by the non-degeneracy of the polarization. Similarly,

$$(\alpha, (C'C)^{\dagger}\gamma) = (C'C\alpha, \gamma) = (C\alpha, C'^{\dagger}\gamma) = (\alpha, C^{\dagger}C'^{\dagger}\gamma)$$

gives the third assertion. □

The following formal property will play an important role in the final part of our argument. It appeared in [69], Lemma 5. We recall that the restriction of a polarization on a Hodge structure to a sub-Hodge structure is non-degenerate (*cf.* Remark 1.3.6).

Proposition 1.3.10 (Invariance of rank)**.** *Let X, Y, $\mathscr{Z}$, C as above. We have*

$$\mathrm{rank}(C) = \mathrm{rank}(C^{\dagger}) = \mathrm{rank}(CC^{\dagger}) = \mathrm{rank}(C^{\dagger}C).$$

In particular,

$$\mathrm{Ker}(CC^{\dagger}) = \mathrm{Ker}(C^{\dagger});$$

$$\mathrm{Ker}(C^{\dagger}C) = \mathrm{Ker}(C);$$

$$\mathrm{Im}(CC^{\dagger}) = \mathrm{Im}(C);$$

$$\mathrm{Im}(C^{\dagger}C) = \mathrm{Im}(C^{\dagger}).$$

Proof. We only need to show $\mathrm{Ker}(C^{\dagger}C) = \mathrm{Ker}(C)$. Indeed, replacing C by $C^{\dagger}$ gives another equality for kernels since $C^{\dagger\dagger} = C$, then combining the obvious fact that $\mathrm{rank}(C) = \mathrm{rank}(C^{\dagger})$ we get all the equalities of ranks, and the equalities of images follow immediately.

Now $\mathrm{Ker}(C^{\dagger}C) \supset \mathrm{Ker}(C)$ is obvious. For the other inclusion, let $\alpha \in \mathrm{Ker}(C^{\dagger}C)$, we have $(C\alpha, C\alpha') = (C^{\dagger}C\alpha, \alpha') = 0$ for any $\alpha' \in H^k(X)$. However, since C is induced by an algebraic correspondence, it is a morphism of Hodge structures; in particular $\mathrm{Im}(C)$ is a sub-Hodge structure, therefore as we remarked above, the restriction $(-,-)|_{\mathrm{Im}(C)}$ is non-degenerate, which implies $C\alpha = 0$, *i.e.* $\alpha \in \mathrm{Ker}(C)$. This gives the other inclusion $\mathrm{Ker}(C^{\dagger}C) \subset \mathrm{Ker}(C)$. □

1.3.3 The proof of the main theorem

We now prove the main theorem which says that Conjecture 1.2.4 is implied by the Lefschetz standard conjecture:

Theorem 1.3.11 (= Theorem 1.0.2)**.** *Let X be a smooth projective variety of dimension n, $0 \leq k \leq n$ be an integer, and $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ be a big cohomology class* [4]. *Let L be the kernel of the following morphism of 'cup product with γ':*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Assuming the Lefschetz standard conjecture, then L is supported on a divisor of X, that is,

$$L \subset \mathrm{Ker}(H^k(X, \mathbf{Q}) \to H^k(X \backslash Z, \mathbf{Q}))$$

for some closed algebraic subset Z of codimension 1.

4. *cf.* Definition 1.2.1

Proof. Firstly, recall that in Proposition 1.2.5 we have constructed a (not necessarily connected) smooth projective variety Y of dimension $k-1$ with a morphism $\mu : Y \to X$, and showed that the composition $L \hookrightarrow H^k(X,\mathbf{Q}) \xrightarrow{\mu^*} H^k(Y,\mathbf{Q})$ is injective, *i.e.* $L \cap \mathrm{Ker}(\mu^*) = \{0\}$.

Note that the Lefschetz standard conjecture on Y tells us the inverse hard Lefschetz isomorphism $(L_Y)^{-1} : H^k(Y,\mathbf{Q}) \xrightarrow{\simeq} H^{k-2}(Y,\mathbf{Q})$ is algebraic. Therefore, the composition

$$C := (L_Y)^{-1} \circ \mu^* : H^k(X,\mathbf{Q}) \to H^{k-2}(Y,\mathbf{Q}) \tag{1.8}$$

is algebraic, *i.e.* $C = [\mathscr{Z}]_*$ for some $\mathscr{Z} \in \mathrm{CH}^{n-1}(X \times Y)_{\mathbf{Q}}$. The above injectivity is of course preserved, thus

$$\mathrm{Ker}(C) \cap L = \{0\}.$$

Taking the orthogonal complements (with respect to the fixed polarization $(-,-)_{H^k(X)}$ introduced in (1.5)) of both sides, and using the non-degeneracy of the polarization, we get:

$$\mathrm{Ker}(C)^{\perp} + L^{\perp} = H^k(X,\mathbf{Q}). \tag{1.9}$$

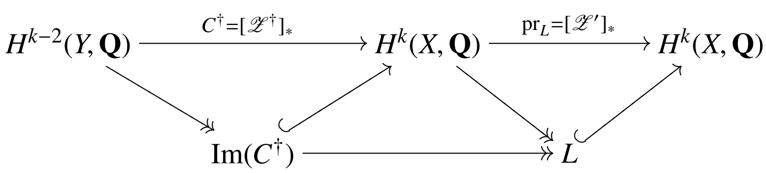
Now consider the adjoint correspondence $C^{\dagger} = [\mathscr{Z}^{\dagger}]_* : H^{k-2}(Y,\mathbf{Q}) \to H^k(X,\mathbf{Q})$, where $\mathscr{Z}^{\dagger} \in \mathrm{CH}_{n-1}(Y \times X)_{\mathbf{Q}}$ is the algebraic cycle constructed in (1.7) using the Lefschetz standard conjecture. By the adjoint property $(C^{\dagger}\alpha, \alpha') = (\alpha, C\alpha')$ for any $\alpha \in H^{k-2}(Y,\mathbf{Q})$ and $\alpha' \in H^k(X,\mathbf{Q})$, we find that $\mathrm{Im}(C^{\dagger}) \subset \mathrm{Ker}(C)^{\perp}$. However, $\dim \mathrm{Ker}(C)^{\perp} = \dim H^k(X) - \dim \mathrm{Ker}(C) = \dim \mathrm{Im}(C) = \dim \mathrm{Im}(C^{\dagger})$, so in fact $\mathrm{Im}(C^{\dagger}) = \mathrm{Ker}(C)^{\perp}$. Therefore (1.9) is equivalent to

$$\mathrm{Im}(C^{\dagger}) + L^{\perp} = H^k(X,\mathbf{Q}). \tag{1.10}$$

We first finish the proof by assuming the following Proposition 1.3.12, which says that with respect to the orthogonal decomposition $H^k(X,\mathbf{Q}) = L \oplus L^{\perp}$ (*cf.* Remark 1.3.6), the orthogonal projector

$$\mathrm{pr}_L : H^k(X,\mathbf{Q}) \twoheadrightarrow L \hookrightarrow H^k(X,\mathbf{Q})$$

is algebraic, *i.e.* induced by an algebraic cycle $\mathscr{Z}' \in \mathrm{CH}^n(X \times X)_{\mathbf{Q}}$.
Now consider the composition of the algebraic correspondences $\mathscr{Z}' \circ \mathscr{Z}^{\dagger}$:



where we place the cohomological correspondences in the first line, and the images of two morphisms in the second line. Then the equality (1.10) says exactly that the induced morphism in the bottom line from $\mathrm{Im}(C^{\dagger})$ to L is surjective, in other words,

$$\mathrm{Im}\Big([\mathscr{Z}' \circ \mathscr{Z}^{\dagger}]_* : H^{k-2}(Y,\mathbf{Q}) \to H^k(X,\mathbf{Q})\Big) = L.$$

Therefore, L is supported on $Z := \mathrm{Supp}\Big(\mathrm{pr}_2(\mathscr{Z}' \circ \mathscr{Z}^{\dagger})\Big)$: the support of the image of $\mathscr{Z}' \circ \mathscr{Z}^{\dagger} \in \mathrm{CH}_{n-1}(Y \times X)_{\mathbf{Q}}$ under the projection to X, so the dimension of each irreducible component of Z is at most $n-1$, hence L is supported on a divisor of X. □

To complete the proof, we only need to show the following

Proposition 1.3.12 (Orthogonal projector to L)**.** *Let X, γ, L as in the above theorem. Assume the Lefschetz standard conjecture holds, then for the orthogonal decomposition*[5] $H^k(X,\mathbf{Q}) = L \oplus L^\perp$ *with respect to the fixed polarization* $(-,-)_{H^k(X)}$*, the orthogonal projector*

$$\mathrm{pr}_L : H^k(X,\mathbf{Q}) \twoheadrightarrow L \hookrightarrow H^k(X,\mathbf{Q})$$

is algebraic (in the sense of Remark 1.1.4).

Proof. Define $B : H^k(X,\mathbf{Q}) \to H^k(X,\mathbf{Q})$ to be the unique morphism satisfying

$$(B\alpha, \alpha')_{H^k(X)} = \int_X \gamma\alpha\alpha'$$

for any $\alpha, \alpha' \in H^k(X)$. (Here the rationality of B comes from those of γ and s_X.)
However, by

$$\begin{aligned} \int_X \gamma\alpha\alpha' &= \int_X L_X^{n-k} \cdot (L_X^{n-k})^{-1} \circ s_X(\gamma\alpha) \cdot s_X(\alpha') \\ &= ((L_X^{n-k})^{-1} \circ s_X(\gamma\alpha), \alpha'), \end{aligned}$$

we deduce that

$$B = (L_X^{n-k})^{-1} \circ s_X \circ (\gamma\cup),$$

where γ is an algebraic class, s_X and $(L_X^{n-k})^{-1}$ are also induced by algebraic correspondences under the assumption of Lefschetz standard conjecture, therefore B is algebraic, *i.e.* $B = [\mathscr{W}]_*$, for some $\mathscr{W} \in \mathrm{CH}^n(X \times X)_{\mathbf{Q}}$.

Observe that $(B\alpha, \alpha') = \int_X \gamma\alpha\alpha' = (-1)^k \int_X \gamma\alpha'\alpha = (-1)^k(B\alpha', \alpha) = (\alpha, B\alpha')$, which means that B is *self-adjoint*:

$$B^\dagger = B.$$

By Proposition 1.3.10, we have $\mathrm{rank}(B^2) = \mathrm{rank}(BB^\dagger) = \mathrm{rank}(B)$, *i.e.* ,

$$\mathrm{Im}(B) = \mathrm{Im}(B^2). \tag{1.11}$$

The following elementary lemma in linear algebra allows us to construct from such an endomorphism a projector onto its image.

Lemma 1.3.13. *Let V be a finite dimensional $\mathbf{Q}$-vector space, $f : V \to V$ be an endomorphism satisfying* $\mathrm{Im}(f^2) = \mathrm{Im}(f)$*. Then there exists a $\mathbf{Q}$-coefficient polynomial P with* $P(0) = 0$*, such that the endomorphism $g := P(f)$ is a projector onto* $\mathrm{Im}(f)$*, i.e.* $g^2 = g$ *and* $\mathrm{Im}(g) = \mathrm{Im}(f)$*. Moreover,* $\mathrm{Ker}(f) = \mathrm{Ker}(g)$.

Proof. By assumption $f|_{\mathrm{Im}(f)} : \mathrm{Im}(f) \to \mathrm{Im}(f)$ is surjective hence an isomorphism. Let $Q \in \mathbf{Q}[T]$ be the minimal polynomial of $f|_{\mathrm{Im}(f)}$, since $f|_{\mathrm{Im}(f)}$ is an isomorphism, $Q(0) \neq 0$.
Defining $R \in \mathbf{Q}[T]$ to be $R[T] = -\frac{Q(T)-Q(0)}{Q(0)\cdot T}$, then $R\left(f|_{\mathrm{Im}(f)}\right) = \left(f|_{\mathrm{Im}(f)}\right)^{-1}$; in other words,

$$(R(f) \cdot f)\,|_{\mathrm{Im}(f)} = \mathrm{id}_{\mathrm{Im}(f)}\,.$$

Now we set $P \in \mathbf{Q}[T]$ to be $P(T) = R(T) \cdot T$. Then $P(0) = 0$, and $g := P(f)$ satisfies

$$g|_{\mathrm{Im}(f)} = \mathrm{id}_{\mathrm{Im}(f)}\,. \tag{1.12}$$

However, since $P(0) = 0$, we have $\mathrm{Im}(g) \subset \mathrm{Im}(f)$, thus (1.12) implies $\mathrm{Im}(g) = \mathrm{Im}(f)$ and thus also $g^2 = g$, *i.e.* g is a projector onto $\mathrm{Im}(f)$.
Moreover, by $P(0) = 0$, we have *a priori* $\mathrm{Ker}(f) \subset \mathrm{Ker}(g)$; but f and g have the same image, thus the same rank, we deduce that $\mathrm{Ker}(f) = \mathrm{Ker}(g)$. □

5. See the last point of Remark 1.3.6.

We continue the proof of the Proposition. By (1.11), we can apply the above lemma to B to get a rational coefficient polynomial P with $P(0) = 0$, such that $P(B)$ is a projector onto $\mathrm{Im}(B)$, and $\mathrm{Ker}(P(B)) = \mathrm{Ker}(B)$. Therefore, $P(B)$ and $\mathrm{id} - P(B)$ is a pair of projectors corresponding to the direct sum decomposition

$$H^k(X, \mathbf{Q}) = \mathrm{Im}(B) \oplus \mathrm{Ker}(B).$$

Moreover, we remark that the above direct sum decomposition is in fact *orthogonal* with respect to $(-,-)_{H^k(X)}$: this is an immediate consequence of the self-adjoint property of B.
To conclude, we remark that $L = \mathrm{Ker}(\gamma\cup) = \mathrm{Ker}(B)$, thus $\Delta_X - P(\mathscr{W}) \in \mathrm{CH}^n(X \times X)_{\mathbf{Q}}$ induces on the cohomology $H^k(X, \mathbf{Q})$ the orthogonal projector $\mathrm{id} - P(B)$ onto L, where Δ_X denotes the diagonal class in $X \times X$, and the multiplication in $P(\mathscr{W})$ is given by composition of correspondences (NOT the intersection product).

This finishes the proof of the Proposition 1.3.12 and thus also the proof of the main Theorem 1.3.11. □

1.4 Final Remarks

Remark 1.4.1 (Unconditional results)**.** Our proof of Conjecture 1.2.4 using the standard conjecture is in fact unconditional in some cases. In the following discussion, let X, γ, L be as in the main theorem 1.3.11, and we adopt all the constructions and notations of its proof in the preceding section.

When $k = 0, 1$, there is nothing to prove. The $k = 2$ case reduces to the Lefschetz theorem on (1,1)-classes for $H^2(X, \mathbf{Q})$.

When $k = 3, 4, 5$, recall that the correspondence needed in the proof of the main theorem 1.3.11 is $\mathrm{pr}_L \circ C^\dagger : H^{k-2}(Y, \mathbf{Q}) \to H^k(X, \mathbf{Q})$, and we use Lefschetz standard conjecture on Y to get the algebraicity of $C^\dagger$, and use it on X to get the algebraicity of pr_L. However, by an explicit calculation:

$$\begin{aligned} C^\dagger &= ((L_Y)^{-1} \circ \mu^*)^\dagger && \text{(see (1.8))} \\ &= (\mu^*)^\dagger \circ (L_Y^{-1})^\dagger && \text{(Lemma 1.3.9)} \\ &= ((L_X^{n-k})^{-1} \circ s_X \circ \mu_* \circ s_Y \circ L_Y) \circ L_Y^{-1} && \text{(by (1.7))} \\ &= (L_X^{n-k})^{-1} \circ s_X \circ \mu_* \circ s_Y \end{aligned}$$

we find that we only need the standard conjecture on X and the algebraicity of the morphism $s_Y : H^{k-2}(Y, \mathbf{Q}) \to H^{k-2}(Y, \mathbf{Q})$. While the algebraicity of s_Y on $H^i(Y)$ for $i \leq 3$ is known: firstly s_Y acts as identity on $H^0(Y)$ and $H^1(Y)$, thus is obviously algebraic; as for $H^2(Y)$ (*resp.* $H^3(Y)$), the Lefschetz decomposition has only two factors, and the projector to the primitive factor can be constructed using only the Lefschetz operator and the inverse $H^{2\dim Y}(Y) \xrightarrow{\simeq} H^0(Y)$ (*resp.* $H^{2\dim Y - 1}(Y) \xrightarrow{\simeq} H^1(Y)$), which is also known to be algebraic. In conclusion, our proof works unconditionally when $k = 3, 4, 5$ for X a smooth complete intersection of a product of curves, surfaces, and abelian varieties *etc.* .

Remark 1.4.2 (A reinterpretation by motivated cycles)**.** To get around the standard conjectures and thus obtain some unconditional theories of motives, Y. André [5] introduced the notion of *motivated cycles*, which is a space of cohomology classes fitting in the following inclusions (conjecturally they are all the same):
{classes of cycles}⊂{motivated cycles}⊂{absolute Hodge classes}⊂{Hodge classes}.
Roughly speaking, motivated cycles are constructed from algebraic cycles by adding the cohomology classes of the inverses of hard Lefschetz isomorphisms in the category of smooth projective

varieties with morphisms given by algebraic correspondences. We refer to the original paper *loc.cit.* for more details, and also to [6] Chapter 9, 10 for an introduction.

Now if we considered motivated cycles and motivated correspondences (=motivated cycles in the product spaces) instead of the algebraic ones, we would not have any problem caused by the standard conjectures. In particular, we could define a sub-Hodge structure L of $H^k(X, \mathbf{Q})$ to be of *motivated coniveau* at least c if there exists a motivated correspondence Γ from another smooth projective variety Y to X, such that L is contained in the image of $\Gamma_* : H^{k-2c}(Y) \to H^k(X)$. In this language, our result Theorem 1.3.11 can be reformulated as:

Theorem 1.4.3. *Let X be a smooth projective variety of dimension n, $0 \leq k \leq n$ be an integer, and $\gamma \in H^{2n-2k}(X, \mathbf{Q})$ be a big cohomology class. Let L be the kernel of the following morphism of 'cup product with γ':*

$$\cup\gamma : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q}).$$

Then L is of motivated coniveau at least 1.

Chapter 2

Decomposition of small diagonals and Chow rings of Calabi-Yau complete intersections

Résumé D'une part, pour une intersection complète générale de type Calabi-Yau X, en utilisant une décomposition de la petite diagonale de $X \times X \times X$ modulo l'équivalence rationnelle, on démontre que tout 0-cycle décomposable de degré 0 est rationnellement équivalent à 0, à une torsion près, où un cycle est dit *décomposable* s'il est une combinaison linéaire des intersections des cycles de dimensions strictement positives. D'autre part, par une décomposition similaire de la plus petite diagonale dans une puissance plus grande d'une hypersurface, on obtient aussi certaine dégénérescence du produit d'intersection de son anneau de Chow.

Abstract On the one hand, for a general Calabi-Yau complete intersection X, we establish a decomposition, up to rational equivalence, of the small diagonal in $X \times X \times X$, from which we deduce that any *decomposable* 0-cycle of degree 0 is in fact rationally equivalent to 0, up to torsion. On the other hand, we find a similar decomposition of the smallest diagonal in a higher power of a hypersurface, which provides us an analogous result on the multiplicative structure of its Chow ring.

2.0 Introduction

For a given smooth projective complex algebraic variety X, we can construct very few subvarieties or algebraic cycles of $X \times X$ in an *a priori* fashion. Besides the divisors and the exterior products of two algebraic cycles of each factor, the diagonal $\Delta_X := \{(x, x) \mid x \in X\} \subset X \times X$ is essentially the only one that we can canonically construct in general. Despite of its simplicity, the diagonal in fact contains a lot of geometric information of the original variety. For instance, its normal bundle is the tangent bundle of X; its self-intersection number is its topological Euler characteristic and so on. Besides these obvious facts, we would like to remark that the Bloch-Beilinson-Murre conjecture (*cf.* [13] [55] [45]), which is considered as one of the deepest conjectures in the study of algebraic cycles, claims a conjectural decomposition of the diagonal, up to rational equivalence, as the sum of certain orthogonal idempotent correspondences related to the Hodge structures on its Betti cohomology groups.

The idea of using decomposition of diagonal to study algebraic cycles is initiated by Bloch and Srinivas [18]; we state their main theorem in the following form.

Theorem 2.0.1 (Bloch, Srinivas [18])**.** *Let X be a smooth projective complex algebraic variety of dimension n. Suppose that $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset Y, i.e. the natural morphism $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then there exist a positive integer $m \in \mathbf{N}^*$ and a proper closed algebraic subset $D \subsetneq X$, such that in $\mathrm{CH}_n(X \times X)$ we have*

$$m \cdot \Delta_X = \mathscr{Z}_1 + \mathscr{Z}_2 \tag{2.1}$$

where $\mathscr{Z}_1$ is supported on $Y \times X$, and $\mathscr{Z}_2$ is supported on $X \times D$.

The above decomposition in the case that Y is a point, or equivalently $\mathrm{CH}_0(X) \simeq \mathbf{Z}$ by degree map, is further generalized by Paranjape [59] and Laterveer [49] for varieties with small Chow groups in the following form.

Theorem 2.0.2 (Paranjape [59], Laterveer [49])**.** *Let X be a smooth projective n-dimensional variety. If the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbf{Q}} \to H^{2n-2i}(X, \mathbf{Q})$ is injective for any $0 \leq i \leq c-1$. Then there exist a positive integer $m \in \mathbf{N}^*$, a closed algebraic subset T of dimension $\leq n-c$, and for each $i \in \{0, 1, \cdots, c-1\}$, a pair of closed algebraic subsets V_i, W_i with $\dim V_i = i$ and $\dim W_i = n-i$, such that in $\mathrm{CH}_n(X \times X)$, we have*

$$m \cdot \Delta_X = \mathscr{Z}_0 + \mathscr{Z}_1 + \cdots + \mathscr{Z}_{c-1} + \mathscr{Z}' \tag{2.2}$$

where $\mathscr{Z}_i$ is supported on $V_i \times W_i$ for any $0 \leq i < c$, and $\mathscr{Z}'$ is supported on $X \times T$.

For applications of such decompositions, the point is that we consider (2.1) and (2.2) as equalities of correspondences from X to itself, which yield decompositions of the identity correspondence. This point of view allows us to deduce from (2.1) and (2.2) many interesting results like generalizations of Mumford's theorem (*cf.* [54] [18] [68]).

Most of this chapter is devoted to the study of the class of the *small diagonal*

$$\delta_X := \{(x, x, x) \in X^3 \mid x \in X\} \tag{2.3}$$

in $\mathrm{CH}_n(X^3)_{\mathbf{Q}}$, where X is an n-dimensional Calabi-Yau variety. The interest of the study is motivated by the obvious fact that while the diagonal seen as a self-correspondence of X controls $\mathrm{CH}^*(X)_{\mathbf{Q}}$ as an additive object, the small diagonal seen as a correspondence between $X \times X$ and X controls the multiplicative structure of $\mathrm{CH}^*(X)_{\mathbf{Q}}$.

The first result in this direction is due to Beauville and Voisin [12], who find a decomposition of the *small* diagonal $\delta_S := \{(x, x, x) \mid x \in S\}$ in $\mathrm{CH}_2(S \times S \times S)$ for S an algebraic K3 surface.

Theorem 2.0.3 (Beauville, Voisin [12])**.** *Let S be a projective K3 surface, and $c_S \in \mathrm{CH}_0(S)$ be the well-defined*[1] *0-dimensional cycle of degree 1 represented by any point lying on any rational curve of S. Then in $\mathrm{CH}_2(S \times S \times S)$, we have*

$$\delta_S = \Delta_{12} + \Delta_{13} + \Delta_{23} - S \times c_S \times c_S - c_S \times S \times c_S - c_S \times c_S \times S \tag{2.4}$$

where Δ_{12} is represented by $\{(x, x, c_S) \mid x \in S\}$, and Δ_{13}, Δ_{23} are defined similarly.

As is mentioned above, we regard (2.4) as an equality of correspondences from $S \times S$ to S. Applying this to a product of two divisors $D_1 \times D_2$, one can recover the following corollary, which is in fact a fundamental observation in [12] for the proof of the above theorem.

1. The fact that c_S is well-defined relies on the result of Bogomolov-Mumford (*cf.* the appendix of [52]) about the existence of rational curves in any ample linear system, *cf.* [12].

Corollary 2.0.4 (Beauville, Voisin [12]). *Let S be a projective K3 surface. Then the intersection product of any two divisors is always proportional to the class c_S in $\mathrm{CH}_0(S)$, i.e.*

$$\mathrm{Im}\Big(\mathrm{Pic}(S)\times\mathrm{Pic}(S)\xrightarrow{\bullet}\mathrm{CH}_0(S)\Big)=\mathbf{Z}\cdot c_S.$$

As is pointed out in [12], this corollary is somehow surprising because $\mathrm{CH}_0(S)$ is *a priori* very huge ('of infinite dimension' in the sense of Mumford, *cf.* [54]).

The next result in this direction, which is also the starting point of this chapter, is the following partial generalization of Theorem 2.0.3 due to Voisin:

Theorem 2.0.5 (Voisin [75]). *Let $X\subset\mathbf{P}^{n+1}$ be a general smooth hypersurface of Calabi-Yau type, that is, the degree of X is $n+2$. Let $h:=c_1(\mathscr{O}_X(1))\in\mathrm{CH}^1(X)$ be the hyperplane section class, $h_i:=\mathrm{pr}_i^*(h)\in\mathrm{CH}^1(X^3)$ for $i=1,2,3$, and $c_X:=\frac{h^n}{n+2}\in\mathrm{CH}_0(X)_{\mathbf{Q}}$ be a $\mathbf{Q}$-0-cycle of degree 1. Then we have a decomposition of the small diagonal in $\mathrm{CH}_n(X^3)_{\mathbf{Q}}$*

$$\delta_X=\frac{1}{(n+2)!}\Gamma+\Delta_{12}+\Delta_{13}+\Delta_{23}+P(h_1,h_2,h_3),\tag{2.5}$$

where $\Delta_{12}=\Delta_X\times c_X$, and Δ_{13}, Δ_{23} are defined similarly; P is a homogeneous polynomial of degree $2n$; and $\Gamma:=\bigcup_{t\in F(X)}\mathbf{P}_t^1\times\mathbf{P}_t^1\times\mathbf{P}_t^1\subset X^3$, where $F(X)$ is the variety of lines of X, and $\mathbf{P}_t^1$ is the line corresponding to $t\in F(X)$.

Applying (2.5) as an equality of correspondences, she deduces the following

Corollary 2.0.6 (Voisin [75]). *In the same situation as the above theorem, the intersection product of any two cycles of complementary and strictly positive codimensions is always proportional to c_X in $\mathrm{CH}_0(X)_{\mathbf{Q}}$, i.e. for any $i,j\in\mathbf{N}^*$ with $i+j=n$, we have*

$$\mathrm{Im}\Big(\mathrm{CH}^i(X)_{\mathbf{Q}}\times\mathrm{CH}^j(X)_{\mathbf{Q}}\xrightarrow{\bullet}\mathrm{CH}_0(X)_{\mathbf{Q}}\Big)=\mathbf{Q}\cdot c_X.$$

In particular, for any $i=1,\cdots,m$, let Z_i,Z_i' be algebraic cycles of strictly positive codimension with $\dim Z_i+\dim Z_i'=n$, then any equality on the cohomology level $\sum_{i=1}^m\lambda_i[Z_i]\cup[Z_i']=0$ in $H_0(X,\mathbf{Q})$ is in fact an equality modulo rational equivalence: $\sum_{i=1}^m\lambda_iZ_i\bullet Z_i'=0$ in $\mathrm{CH}_0(X)_{\mathbf{Q}}$.

The main results of this chapter are further generalizations of Voisin's theorem and its corollary in two different directions.

The first direction of generalization is about smooth Calabi-Yau complete intersections in projective spaces:

Theorem 2.0.7 (=Theorem 2.1.12+Theorem 2.1.15+Theorem 2.1.17+Theorem 2.1.18). *Let E be a rank r vector bundle on $\mathbf{P}^{n+r}$ satisfying the following positivity condition:*
($*$) *The evaluation map $H^0(\mathbf{P}^{n+r},E)\to\oplus_{i=1}^3E_{y_i}$ is surjective for any three distinct collinear points $y_1,y_2,y_3\in\mathbf{P}^{n+r}$.*
Let X be a smooth n-dimensional subvariety of $\mathbf{P}^{n+r}$, which is the zero locus of a section of E. Suppose that the canonical bundle of X is trivial, i.e. $\det(E)\simeq\mathscr{O}_{\mathbf{P}^{n+r}}(n+r+1)$. Then we have:

(i) There are symmetric homogeneous polynomials with $\mathbf{Z}$-coefficients[2] Q,P, such that in $\mathrm{CH}_n(X^3)$:

$$a_0\deg(X)\cdot\delta_X=\Gamma+j_{12*}(\mathscr{Z})+j_{13*}(\mathscr{Z})+j_{23*}(\mathscr{Z})+P(h_1,h_2,h_3)$$

2. See Theorem 2.1.12 for more about their coefficients.

$$\text{with} \quad \mathscr{Z} := Q(h_1, h_2) \quad \text{in } \mathrm{CH}_n(X \times X),$$

where Γ *is defined to be the virtual fundamental class defined in (2.7) and (2.8), (it is as in Theorem 2.0.5 if* X *is generic);* $h_i \in \mathrm{CH}^1(X^3)$ *or* $\mathrm{CH}^1(X^2)$ *is the pull-back of* $h = c_1(\mathscr{O}_X(1))$ *by the* i^{th} *projection; and the inclusions of big diagonals are given by:*

$$\begin{array}{rll} & X^2 & \hookrightarrow X^3 \\ j_{12}: & (x, x') & \mapsto (x, x, x') \\ j_{13}: & (x, x') & \mapsto (x, x', x) \\ j_{23}: & (x, x') & \mapsto (x', x, x). \end{array}$$

(ii) If the coefficients of Q*, determined by (2.20) in Proposition 2.1.9 and Lemma 2.1.11 satisfy* $a_0 \neq 0$ *and* $a_1 \neq a_0$*, then the intersection product of two cycles of strictly positive complementary codimension is as simple as possible: for any* $k, l \in \mathbf{N}^*$ *with* $k + l = n$,

$$\mathrm{Im}\left(\bullet : \mathrm{CH}^k(X)_{\mathbf{Q}} \times \mathrm{CH}^l(X)_{\mathbf{Q}} \to \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n,$$

where $h = c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X)$.

(iii) The conditions $a_0 \neq 0$ *and* $a_1 \neq a_0$ *are always satisfied in the splitting case, that is,* $E = \oplus_{i=1}^r \mathscr{O}_{\mathbf{P}}(d_i)$ *with* $d_i \geq 2$*, hence the conclusions of (i) and (ii) stated above hold for for any smooth Calabi-Yau complete intersections* [3].

A word for the conditions $a_0 \neq 0$ and $a_1 \neq a_0$ appeared above: they are (mild) numerical conditions on the Chern numbers of the vector bundle E, which are satisfied for complete intersections (see (iii)). While for the full generality as in (i) and (ii), it seems that we have to add them, but presumably they are automatic under the positivity condition $(*)$ (see Remark 2.1.20).

Again, the above degeneration property of the intersection product is remarkable since CH_0 is very huge. To emphasize the principle that the small diagonal controls the intersection product, we point out that part (ii) of the preceding theorem is obtained by applying the equality of part (i) as correspondences to an exterior product of two algebraic cycles.

We will make a comparison of our result (iii) for Calabi-Yau complete intersections with Beauville's 'weak splitting principle' in [10] for holomorphic symplectic varieties, which says [4] that any polynomial relation between the *cohomological* Chern classes of lines bundles and the tangent bundle holds already for their *Chow-theoretical* Chern classes. In our case we prove more: any *decomposable* **Q**-coefficient 0-cycle is rational equivalent to 0 if and only if it has degree 0, see Remark 2.1.19.

We also give in §2.1.5 an example of a surface S in $\mathbf{P}^3$ of general type, such that

$$\mathrm{Im}\left(\bullet : \mathrm{CH}^1(S)_{\mathbf{Q}} \times \mathrm{CH}^1(S)_{\mathbf{Q}} \to \mathrm{CH}_0(S)_{\mathbf{Q}}\right) \supsetneqq \mathbf{Q} \cdot h^2,$$

where $h = c_1(\mathscr{O}_S(1)) \in \mathrm{CH}^1(S)$, in contrast to the result for K3 surfaces proved in [12]. This example supports the feeling that the Calabi-Yau condition gives some strong restrictions on the multiplicative structure of the Chow ring.

The second direction of generalization is about higher powers (≥ 3) of hypersurfaces with ample or trivial canonical bundle. The objective is to decompose the smallest diagonal in a higher

3. See Theorem 2.1.17 and Theorem 2.1.18 for the precise statements.
4. This is the strengthened version conjectured by Voisin in [71].

self-product, and deduce from it some implication on the multiplicative structure of the Chow ring of the variety. That such a decomposition should exist was suggested by Nori to be the natural generalization of Theorem 2.0.5. We also refer to [38] for similar results in the case of curves. Now we state our result precisely:

Theorem 2.0.8 (=Theorem 2.2.12 + Theorem 2.2.13)**.** *Let X be a smooth hypersurface in $\mathbf{P}^{n+1}$ of degree d with $d \geq n+2$. Let $k = d+1-n \geq 3$. Then*

(i) One of the following two cases occurs:

(a) There exist rational numbers λ_j for $j = 2, \cdots, k-1$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\mathrm{CH}_n(X^k)_{\mathbf{Q}}$ we have:

$$\delta_X = (-1)^{k-1}\frac{1}{d!}\cdot\Gamma + \sum_{i=1}^{k} D_i + \sum_{j=2}^{k-2}\lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_k),$$

where Γ is the virtual fundamental class defined in a similar way as in Theorem 2.0.7, and the bigger diagonals D_I are defined in (2.39) or (2.40), and $D_i := D_{\{i\}}$.
Or

(b) There exist a (smallest) integer $3 \leq l < k$, rational numbers λ_j for $j = 2, \cdots, l-2$, and a symmetric homogeneous polynomial P of degree $n(l-1)$, such that in $\mathrm{CH}_n(X^l)_{\mathbf{Q}}$ we have:

$$\delta_X = \sum_{i=1}^{l} D_i + \sum_{j=2}^{l-2}\lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_l).$$

Moreover, $\Gamma = 0$ if $d \geq 2n$.

(ii) For any strictly positive integers $i_1, i_2, \cdots, i_{k-1} \in \mathbf{N}^*$ with $\sum_{j=1}^{k-1} i_j = n$, the image

$$\mathrm{Im}\Big(\mathrm{CH}^{i_1}(X)_{\mathbf{Q}} \times \mathrm{CH}^{i_2}(X)_{\mathbf{Q}} \times \cdots \times \mathrm{CH}^{i_{k-1}}(X)_{\mathbf{Q}} \xrightarrow{\bullet} \mathrm{CH}_0(X)_{\mathbf{Q}}\Big) = \mathbf{Q}\cdot h^n$$

We remark that when $d > n+2$, part (ii) of the previous theorem is in fact implied by the Bloch-Beilinson conjecture, see Remark 2.2.14.

The main line of the proofs of the above theorems is the same as in Voisin's paper [75]: one proceeds in three steps:

- Firstly, one 'decomposes' the class of Γ (see Theorem 2.0.5 for its definition) restricting to the complementary of the small diagonal, by means of a careful study of the geometry of collinear points on the variety.

- Secondly, by the localization exact sequence for Chow groups, we obtain a decomposition of certain multiple of the small diagonal in terms of Γ and other cycles of diagonal type or coming from the ambient space.

- Thirdly, once we have a decomposition of small diagonal, regarded as an equality of correspondences, we will draw consequences on the multiplicative structure of the Chow ring.

We remark that at some point of the proof, one should verify that the 'multiple' appeared in the decomposition is non-zero to get a genuine decomposition of the small diagonal, and also some coefficients should be distinct to deduce the desired conclusion on the multiplicative structure of

Chow rings. These are too easy to be noticed in [75], but become the major difficulties in our present chapter.

The chapter consists of two parts. The first part deals with the first direction of generalization explained above (Theorem 2.0.7), where we start by the geometry of collinear points to deduce a decomposition of a certain multiple of the small diagonal; then we deduce from it our result on the multiplicative structure of Chow rings; after that we treat the complete intersection case to obtain the main results; finally we construct an example of surface in $\mathbf{P}^3$ such that the image of intersection product of line bundles is 'non-trivial', in contrast to the Calabi-Yau case. The second part establishes Theorem 2.0.8, the second direction of generalization, and we also follow the line of geometry of collinear points, decomposition of the smallest diagonal, and finally consequence on Chow ring's structure.

We will work over the complex numbers throughout this chapter for simplicity, but all the results and proofs go through for any uncountable algebraic closed field of characteristic zero.

2.1 Calabi-Yau complete intersections

The main goal of this section is to prove Theorem 2.0.7 in the introduction. First of all, we will set up the basic situation. Let E be a rank r vector bundle on the complex projective space $\mathbf{P} := \mathbf{P}^{n+r}$. We always make the following

Positivity Assumption $(*)$: The evaluation map $H^0(\mathbf{P}, E) \to \oplus_{i=1}^3 E_{y_i}$ is surjective for any three collinear points y_1, y_2, y_3 in $\mathbf{P}$, where E_y means the fiber of E over y.

We note that this condition implies in particular:
$(*')$ E is globally generated.
$(*'')$ The restriction of $E(-2) := E \otimes \mathscr{O}_{\mathbf{P}}(-2)$ to each line is globally generated [5].

Let $d \in \mathbf{N}^*$ be such that $\det(E) = \mathscr{O}_{\mathbf{P}}(d)$. Let $f \in H^0(\mathbf{P}^{n+r}, E)$ be a global section of E, such that the subscheme of $\mathbf{P}$ defined by f, denoted by

$$X := V(f) \subset \mathbf{P}^{n+r},$$

is smooth of dimension n.

Remark 2.1.1. In the case that f is generic, X is smooth of expected dimension n. Indeed, consider the incidence variety

$$I := \left\{([f], x) \in \mathbf{P}\left(H^0(\mathbf{P}^{n+r}, E)\right) \times \mathbf{P}^{n+r} \mid f(x) = 0\right\}.$$

Let $q : I \to \mathbf{P}\left(H^0(\mathbf{P}^{n+r}, E)\right)$ and $p : I \to \mathbf{P}^{n+r}$ be the two natural projections. The global generated property of E implies that p is a projective bundle, thus I is smooth of dimension $h^0(\mathbf{P}^{n+r}, E) - 1 + n$. Therefore the theorem of generic smoothness applied to q proves the assertion.

We are interested in the case when X is of *Calabi-Yau type*: $K_X = 0$, or equivalently,

Calabi-Yau Assumption: $d = n + r + 1$.

Throughout this section, we will always work in the above setting. A typical example of such situation is when $E = \bigoplus_{i=1}^r \mathscr{O}_{\mathbf{P}}(d_i)$ with $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$, X is hence a smooth Calabi-Yau complete intersection of multi-degree $(d_1, \cdots, d_r)$ and $d = \sum_{i=1}^r d_i = n + r + 1$. Since the case of K3 surfaces is well treated in [12], we assume $n \geq 3$ from now on.

5. Equivalently speaking, along any line $\mathbf{P}^1$, the splitting type $E|_{\mathbf{P}^1} = \oplus_{i=1}^r \mathscr{O}_{\mathbf{P}^1}(d_i)$ satisfies $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$.

2.1.1 Decomposition of small diagonals

Like in the paper [75], our strategy is to express the class of the small diagonal by investigating the lines contained in X. The decomposition result is Corollary 2.1.10 and see Theorem 2.1.12 for its more precise form.

Let $G := \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^{n+r})$ be the Grassmannian of projective lines in $\mathbf{P}$, and for a point $t \in G$, we denote by $\mathbf{P}^1_t$ the corresponding line. Define the *variety of lines* of X:

$$F(X) := \left\{ t \in G \mid \mathbf{P}^1_t \subset X \right\}.$$

More accurately, let $p : L \to G$ be the universal projective line over the Grassmannian, and $q : L \to \mathbf{P}$ be the natural morphism. Then the section f of E gives rise to a section σ_f of the vector bundle $p_* q^* E$ on G, and $F(X)$ is the subscheme of G defined by $\sigma_f = 0$.

Lemma 2.1.2. *If f is general, $F(X)$ is non-empty, smooth and of dimension $n - 3$.*

Proof. Thanks to our positivity assumption $(*'')$, the vector bundle $E|_{\mathbf{P}^1_t}$ has splitting type $\oplus_{i=1}^{r} \mathscr{O}_{\mathbf{P}^1}(d_i)$ satisfying $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$ for any $t \in G$, thus

$$\dim H^0(\mathbf{P}^1_t, E|_{\mathbf{P}^1_t}) = \sum_{i=1}^{r} (d_i + 1) = d + r = n + 2r + 1.$$

The last equality comes from the Calabi-Yau assumption. Therefore by Grauert's base-change theorem, $p_* q^* E$ is a vector bundle of rank $n + 2r + 1$, and thus the *expected* dimension of $F(X)$ is

$$\exp\dim F(X) = \dim G - \mathrm{rank}(p_* q^* E) = 2(n + r - 1) - (n + 2r + 1) = n - 3 \geq 0.$$

Now we can use the results in for example [27] to conclude. □

Still in the case that f is generic, we define

$$\Gamma := \bigcup_{t \in F(X)} \mathbf{P}^1_t \times \mathbf{P}^1_t \times \mathbf{P}^1_t \subset X^3. \tag{2.6}$$

It is then an n-dimensional subvariety of X^3.

When f is not assumed to be general, as long as X is smooth of the expected dimension, we can always define, in a purely intersection theoretical way, the *virtual fundamental classes* of $F(X)$ and Γ as algebraic cycles of the expected dimensions modulo rational equivalence, which coincide with their fundamental classes if f is general. More precisely,

$$F(X)^{vir} := c_{n+2r+1}(p_* q^* E) \in \mathrm{CH}_{n-3}(G), \tag{2.7}$$

where the morphisms are defined in the following diagram, and S is the universal rank 2 vector bundle on G.

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow[q]{} & \mathbf{P} \\ {\scriptstyle p}\downarrow & & \\ G & & \end{array}$$

By the theory of *localized top Chern class* (*cf.* [34, §14.1]), $F(X)^{vir}$ is a cycle supported on the subscheme $F(X)$.

Similarly, we define

$$\Gamma^{vir} := q^3_* p^{3*}(F(X)^{vir}) \in \mathrm{CH}_n(X^3) \tag{2.8}$$

where the morphisms are defined in the following diagram:

$$\begin{array}{ccc} \mathbf{P}(S|_{F(X)})^{\times_{F(X)}3} & \xrightarrow{q^3} & X^3 \\ {\scriptstyle p^3}\downarrow & & \\ F(X) & & \end{array}$$

From now on, we will only be interested in the *virtual* fundamental class of Γ defined in this way, which is an n-dimensional algebraic cycle of X^3, and to simplify the notation, we will write Γ for Γ^{vir} in the rest of the chapter.

To get a decomposition of the small diagonal, we first decompose or calculate the class of Γ_o in $\mathrm{CH}_n(X^3\backslash\delta_X)$, where Γ_o is the restriction of Γ to $X^3\backslash\delta_X$.

Before doing so, let us make some preparatory geometric constructions. Let $\delta_{\mathbf{P}}$ be the small diagonal of $\mathbf{P}^{\times 3} := \mathbf{P}\times\mathbf{P}\times\mathbf{P}$. Define the following closed subvariety of $\mathbf{P}^{\times 3}\backslash\delta_{\mathbf{P}}$:

$$W := \left\{(y_1, y_2, y_3) \in \mathbf{P}^{\times 3} \mid y_1, y_2, y_3 \text{ are collinear}\right\}\backslash\delta_{\mathbf{P}}.$$

In other words, if we denote by $L \to G$ the universal line over the Grassmannian G of projective lines, then in fact $W = L\times_G L\times_G L\backslash\delta_L$. In particular W is a smooth variety with

$$\dim W = \dim G + 3 = 2n + 2r + 1.$$

Similarly, let δ_X be the small diagonal of X^3. We define a closed subvariety V of $X^3\backslash\delta_X$ by taking the closure of $V_o := \left\{(x_1, x_2, x_3) \in X^3 \mid x_1, x_2, x_3 \text{ are collinear and distinct}\right\}$ in $X^3\backslash\delta_X$:

$$V := \overline{V_o}.$$

We remark that the boundary $\partial V := V\backslash V_o$ consists, up to a permutation of the three coordinates, of points of the form (x, x, x') with $x \neq x'$ such that the line joining x, x' is tangent to X at x.
We will also need the 'big' diagonals in $X^3\backslash\delta_X$:

$$\Delta_{12} := \left\{(x, x, x') \in X^3 \mid x \neq x'\right\},$$

and Δ_{13}, Δ_{23} are defined in the same way.

Lemma 2.1.3. *Consider the intersection of W and $X^3\backslash\delta_X$ in $\mathbf{P}^{\times 3}\backslash\delta_{\mathbf{P}}$. The intersection scheme has four irreducible components:*

$$W \cap (X^3\backslash\delta_X) = V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}. \tag{2.9}$$

The intersection along V is transversal, in particular $\dim V = 2n-r+1$. *The intersection along Δ_{ij} is not proper, having excess dimension $r-1$, but the multiplicity of Δ_{ij} in the intersection scheme is 1, where $1 \leq i < j \leq 3$. In particular, the intersection scheme is reduced and the above identity (2.9) also holds scheme-theoretically:*

$$\begin{array}{ccc} V\cup\Delta_{12}\cup\Delta_{13}\cup\Delta_{23} & \longrightarrow & X^3\backslash\delta_X \\ \downarrow & \square & \downarrow \\ W & \longrightarrow & \mathbf{P}^{\times 3}\backslash\delta_{\mathbf{P}} \end{array}$$

Proof. It is obvious that (2.9) holds set-theoretically. To verify (2.9) scheme-theoretically, let

$$W_o := \left\{(y_1, y_2, y_3) \in \mathbf{P}^{\times 3} \mid y_1, y_2, y_3 \text{ are collinear and distinct}\right\}.$$

Consider the incidence variety

$$I := \left\{([f], (y_1, y_2, y_3)) \in \mathbf{P}\left(H^0(\mathbf{P}^{n+r}, E)\right) \times W_o \mid f(y_1) = f(y_2) = f(y_3) = 0\right\}.$$

Let $q : I \to \mathbf{P}\left(H^0(\mathbf{P}^{n+r}, E)\right)$ and $p : I \to W_o$ be the two natural projections. The positivity assumption $(*)$ means precisely that p is a $\mathbf{P}^{h^0(\mathbf{P},E)-1-3r}$-bundle, therefore I is smooth of dimension $h^0(\mathbf{P}, E) + 2n - r$. Since for general f, the corresponding variety X contains a line (Lemma 2.1.2), q is of dominant. By the theorem of generic smoothness, the fiber of q, which is exactly V_o, is smooth of dimension $2n - r + 1$. In particular, V_o is reduced, of locally complete intersection in W_o of codimension $3r$ as expected. In other words, the intersection is transversal along a general point [6] of V.

The assertions concerning the big diagonals are easier: by passing to a general point of Δ_{ij}, it amounts to prove that the intersection scheme of $\Delta_{\mathbf{P}}$ and X^2 in $\mathbf{P}^{\times 2}$ is Δ_X with multiplicity 1, (and of excess dimension r). □

Now we construct a vector bundle F on W. Let S be the tautological rank 2 vector bundle on W, with fiber $S_{y_1y_2y_3}$ over a point $(y_1, y_2, y_3) \in W$ the 2-dimensional vector space corresponding to the projective line $\mathbf{P}^1_{y_1y_2y_3}$ determined by these three collinear points. Therefore $p : \mathbf{P}(S) \to W$ is the $\mathbf{P}^1$-bundle of universal line, and it admits three tautological sections $\sigma_i : W \to \mathbf{P}(S)$ determined by the points y_i, where $i = 1, 2, 3$. Let $q : \mathbf{P}(S) \to \mathbf{P}^{n+r}$ be the natural morphism. We summarize the situation by the following diagram:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{q} & \mathbf{P} \\ {\scriptstyle \sigma_i}\uparrow\uparrow\uparrow \ \downarrow{\scriptstyle p} & & \\ W & & \end{array}$$

Let D_i be the image of section σ_i, which is a divisor of $\mathbf{P}(S)$, for $i = 1, 2, 3$. We define the following sheaf on W:

$$F := p_*(q^*E \otimes \mathscr{O}_{\mathbf{P}(S)}(-D_1 - D_2 - D_3)) \tag{2.10}$$

Lemma 2.1.4. *F is a vector bundle on W of rank $n - r + 1$, with fiber*

$$F_{y_1y_2y_3} = H^0(\mathbf{P}^1_{y_1y_2y_3}, E|_{\mathbf{P}^1_{y_1y_2y_3}} \otimes \mathscr{O}(-y_1 - y_2 - y_3)).$$

Proof. For any $(y_1, y_2, y_3) \in W$, it is obvious that the restriction of the vector bundle $q^*E \otimes \mathscr{O}(-D_1 - D_2 - D_3)$ to the fiber $p^{-1}(y_1, y_2, y_3) =: \mathbf{P}^1_{y_1y_2y_3}$ is exactly $E|_{\mathbf{P}^1_{y_1y_2y_3}} \otimes \mathscr{O}(-y_1 - y_2 - y_3)$. By the positivity assumption $(*)$, the splitting type of E at $\mathbf{P}^1_{y_1y_2y_3}$ is $\oplus_{i=1}^r \mathscr{O}(a_i)$ with $a_1 \geq a_2 \geq \cdots \geq a_r \geq 2$, we find that

$$h^0\left(\mathbf{P}^1_{y_1y_2y_3}, E|_{\mathbf{P}^1_{y_1y_2y_3}} \otimes \mathscr{O}(-y_1 - y_2 - y_3)\right) = h^0\left(\mathbf{P}^1, \oplus_{i=1}^r \mathscr{O}(a_i - 3)\right) = \sum_{i=1}^r (a_i - 2) = d - 2r = n - r + 1,$$

which is independent of the point of W. Now the lemma is a consequence of Grauert's base-change theorem. □

6. This suffices for the scheme-theoretical assertions concerning V, since we work over the complex numbers, there are enough (closed) points such that any algebraic condition satisfied by a general point is also satisfied by the generic point.

Here is the motivation to introduce the vector bundle F: the section $f \in H^0(\mathbf{P}, E)$ gives rise to a section $s_f \in H^0(V, F|_V)$ in the way that for any $(x_1, x_2, x_3) \in V$ the value $s_f(x_1, x_2, x_3)$ is simply given by $f|_{\mathbf{P}^1_{x_1x_2x_3}} \in F_{x_1x_2x_3} = H^0(\mathbf{P}^1_{x_1x_2x_3}, E|_{\mathbf{P}^1_{x_1x_2x_3}} \otimes \mathscr{O}(-x_1 - x_2 - x_3))$ (see Lemma 2.1.4), because f vanishes on x_i by definition. As a result,

Lemma 2.1.5. $c_{n-r+1}(F|_V) = \Gamma_o \in \mathrm{CH}_n(X^3 \backslash \delta_X)$, *where* Γ_o *is the restriction of the variety* Γ *constructed in (2.6) to the open subset* $X^3 \backslash \delta_X$.

Proof. By construction, Γ_o is exactly the zero locus of the section s_f of $F|_V$. By Lemma 2.1.2, Γ_o is n-dimensional, thus represents the top Chern class of $F|_V$. □

Now consider the cartesian diagram (Lemma 2.1.3):

$$\begin{array}{ccc} V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \xrightarrow{i_2} & X^3 \backslash \delta_X \\ \downarrow{\scriptstyle i_4} & \square & \downarrow{\scriptstyle i_1} \\ W & \xrightarrow[i_3]{} & \mathbf{P}^{\times 3} \backslash \delta_{\mathbf{P}} \end{array} \tag{2.11}$$

Since i_1 is clearly a regular embedding, we can apply the theory of refined Gysin maps of [34] to the cycle $c_{n-r+1}(F) \in \mathrm{CH}^{n-r+1}(W)$ in the above diagram. Before doing so, recall that in Lemma 2.1.3 we have observed that the intersections along Δ_{ij}'s are not proper. Let us first calculate the excess normal bundles of them.

Lemma 2.1.6. *For any* $1 \le i < j \le 3$, *the excess normal sheaf along* $\Delta_{ij} \backslash V$ *is a rank* $r-1$ *vector bundle isomorphic to a quotient* $\frac{\mathrm{pr}_1^* E|_X}{\mathrm{pr}_1^* \mathscr{O}_X(1) \otimes \mathrm{pr}_2^* \mathscr{O}_X(-1)}$, *where we identify* Δ_{ij} *with* $X \times X \backslash \Delta_X$, *and* pr_i *are the natural projections to two factors.*

Proof. For simplicity, assume $i = 1, j = 2$, and write the inclusion $j : \Delta_{12} = X \times X \backslash \Delta_X \hookrightarrow X^3 \backslash \delta_X$, which sends (x, x') to (x, x, x'), where $x \neq x'$. Now we are in the following situation:

$$\begin{array}{ccc} X \times X \backslash \Delta_X & \xrightarrow{j} & X^3 \backslash \delta_X \\ \downarrow & & \\ \mathbf{P}^{\times 2} \backslash \Delta_{\mathbf{P}} & \overset{i_3''}{\searrow} & \downarrow{\scriptstyle i_1} \\ \downarrow{\scriptstyle i_3'} & & \\ W & \xrightarrow[i_3]{} & \mathbf{P}^{\times 3} \backslash \delta_{\mathbf{P}} \end{array}$$

The normal bundle of j is obviously $\mathrm{pr}_1^* TX$. And the normal bundle of i_3 sits in the exact sequence:

$$0 \to N_{i_3'} \to N_{i_3''} \to N_{i_3} \to 0.$$

The normal bundle of i_3'' is $\mathrm{pr}_1^* T\,\mathbf{P}$. As for the normal bundle of i_3', let us reinterpret i_3' as:

$$\begin{array}{ccc} L \times_G L \backslash \Delta_L & \xrightarrow{i_3'} & L \times_G L \times_G L \backslash \delta_L \\ & \searrow \quad G \quad \swarrow & \end{array}$$

where $L \to G$ is the universal $\mathbf{P}^1$-fibration over the Grassmannian of projective lines G. From this we see that the normal bundle of i_3' is the same as the quotient of the two relative (over G) tangent sheaves, thus the fiber of $N_{i'_3}$ at $(y, y') \in \mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}$ is canonically isomorphic to $T_y\,\mathbf{P}^1_{yy'}$. Therefore at the point $(x, x') \in X \times X\backslash\Delta_X$, the fiber of the excess normal bundle is canonically isomorphic to

$$\frac{N_{i''_3,(x,x')}}{N_{i'_3,(x,x')} + N_{j,(x,x')}} = \frac{T_x\,\mathbf{P}}{T_x\,\mathbf{P}^1_{xx'} + T_x X}.$$

As long as the line $\mathbf{P}^1_{xx'}$ is not tangent to X at x, *i.e.* $(x, x') \notin V$, the sum in the denominator is a direct sum, and the fiber of the excess bundle at this point is canonically isomorphic to the $(r-1)$-dimensional vector space

$$\frac{N_{X/\mathbf{P},x}}{T_x\,\mathbf{P}^1_{xx'}} = \frac{E_x}{\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}\dot{x}, \mathbf{C}\dot{x}')},$$

where $\mathbf{C}\dot{x}$ is the 1-dimensional sub-vector space corresponding to $x \in \mathbf{P}$. Therefore along $\Delta_{ij}\backslash V$, the excess normal bundle is isomorphic to $\frac{\mathrm{pr}_1^* E|_X}{\mathrm{pr}_1^*\mathscr{O}_X(1)\otimes\mathrm{pr}_2^*\mathscr{O}_X(-1)}$. □

Now we consider the Gysin map $i_1^!$ in the diagram (2.11) to get the following.

Proposition 2.1.7. *There exists a symmetric homogeneous polynomial P of degree $2n$ with integer coefficients, such that in $\mathrm{CH}_n(X^3\backslash\delta_X)$,*

$$c_{n-r+1}(F|_V) + j_{12*}(\alpha) + j_{13*}(\alpha) + j_{23*}(\alpha) + P(h_1, h_2, h_3) = 0, \tag{2.12}$$

where $h_i = \mathrm{pr}_i^*(h) \in \mathrm{CH}^1(X^3\backslash\delta_X)$ with $h = c_1(\mathscr{O}_X(1))$, $i = 1, 2, 3$; the cycle α is defined by

$$\alpha = c_{n-r+1}\left(F|_{\Delta_{12}}\right)\cdot c_{r-1}\left(\frac{\mathrm{pr}_1^* E|_X}{\mathrm{pr}_1^*\mathscr{O}_X(1)\otimes\mathrm{pr}_2^*\mathscr{O}_X(-1)}\right) \in \mathrm{CH}_n(X^2\backslash\Delta_X); \tag{2.13}$$

and the morphisms $j_{12}, j_{13}, j_{23} : X^2\backslash\Delta_X \hookrightarrow X^3\backslash\delta_X$ are defined by

$$\begin{aligned} j_{12}: &\quad (x, x') \mapsto (x, x, x'); \\ j_{13}: &\quad (x, x') \mapsto (x, x', x); \\ j_{23}: &\quad (x, x') \mapsto (x', x, x). \end{aligned}$$

Proof. By the commutativity of Gysin map and push-forwards ([34] Theorem 6.2(a)):

$$i_{2*}\left(i_1^! c_{n-r+1}(F)\right) = i_1^!\left(i_{3*}c_{n-r+1}(F)\right) \text{ in } \mathrm{CH}_n(X^3\backslash\delta_X). \tag{2.14}$$

While in the right hand side, $i_{3*}c_{n-r+1}(F) \in \mathrm{CH}_{n+3r}(\mathbf{P}^{\times 3}\backslash\delta_{\mathbf{P}}) \simeq \mathrm{CH}_{n+3r}(\mathbf{P}^{\times 3})$, and the Chow ring of $\mathbf{P}^{\times 3}$ is well-known:

$$\mathrm{CH}^*(\mathbf{P}^{\times 3}) = \mathbf{Z}[H_1, H_2, H_3]/(H_i^{n+r+1}; i = 1, 2, 3),$$

where $H_i = \mathrm{pr}_i^*(H)$ with $H \in \mathrm{CH}^1(\mathbf{P})$ being the hyperplane section class. Hence there exists a symmetric homogeneous polynomial P of degree $2n$ with integer coefficients, such that

$$i_{3*}c_{n-r+1}(F) = -P(H_1, H_2, H_3) \text{ in } \mathrm{CH}_{n+3r}(\mathbf{P}^{\times 3}\backslash\delta_{\mathbf{P}}).$$

Combining this with (2.14), and denoting $h_i = H_i|_{X^3} \in \mathrm{CH}^1(X^3)$, we obtain the following equality

$$i_{2*}\left(i_1^! c_{n-r+1}(F)\right) + P(h_1, h_2, h_3) = 0 \text{ in } \mathrm{CH}_n(X^3\backslash\delta_X). \tag{2.15}$$

In the left hand side, by [34] Proposition 6.3, we have:

$$i_1^! c_{n-r+1}(F) = i_1^! \left(c_{n-r+1}(F) \cdot [W]\right) = c_{n-r+1}(F|_{V\cup\Delta_{12}\cup\Delta_{13}\cup\Delta_{23}}) \cdot i_1^!([W]), \tag{2.16}$$

where $[W]$ is the fundamental class of W. Note here $i_1^!([W])$ is a $(2n-r+1)$-dimensional cycle, but $V \cap \Delta_{ij}$ is of dimension strictly less than $2n-r+1$, thus we can use the excess intersection formula ([34] §6.3) component by component in the open subsets $\Delta_{ij}\backslash V$ to get (the excess normal bundle is given in Lemma 2.1.6):

$$i_1^!([W]) = [V] + \sum_{1\leq i<j\leq 3} [\Delta_{ij}] \cdot c_{r-1}\left(\frac{\mathrm{pr}_1^* E|_X}{\mathrm{pr}_1^* \mathscr{O}_X(1) \otimes \mathrm{pr}_2^* \mathscr{O}_X(-1)}\right).$$

Therefore, by omitting all the push-forwards induced by inclusions of subvarieties of $X^3\backslash\delta_X$,

$$i_{2*}\left(c_{n-r+1}(F|_V) \cdot i_1^!([W])\right) = c_{n-r+1}(F|_V);$$

$$i_{2*}\left(c_{n-r+1}(F|_{\Delta_{12}}) \cdot i_1^!([W])\right) = c_{n-r+1}(F|_{\Delta_{12}}) \cdot c_{r-1}\left(\frac{\mathrm{pr}_1^* E|_X}{\mathrm{pr}_1^* \mathscr{O}_X(1) \otimes \mathrm{pr}_2^* \mathscr{O}_X(-1)}\right),$$

putting these in (2.15) and (2.16) we get the desired formula. □

Let us now deal with the equality (2.12) term by term. Firstly, by Lemma 2.1.5, $c_{n-r+1}(F|_V) = \Gamma_o$ in $\mathrm{CH}_n(X^3\backslash\delta_X)$. Secondly, we would like to calculate $F|_{\Delta_{12}}$ of Proposition 2.1.7. We remark that this bundle is the pull-back by the inclusion $X^2\backslash\Delta_X \hookrightarrow \mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}$ of the bundle

$$M := F|_{\Delta_{12,\mathbf{P}}}$$

where $\Delta_{12,\mathbf{P}} = \left\{(y,y,y') \in \mathbf{P}^{\times 3} \mid y \neq y'\right\} \subset W$, and we identify $\Delta_{12,\mathbf{P}}$ with $\mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}$.

We still use S to denote the tautological rank 2 vector bundle on $\mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}$, hence $p : \mathbf{P}(S) \to \mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}$ is the universal line, which admits two tautological sections σ, σ', and we call $q : \mathbf{P}(S) \to \mathbf{P}$ the natural morphism:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{\ q\ } & \mathbf{P} \\ {\scriptstyle \sigma}\uparrow\uparrow{\scriptstyle \sigma'} \ \downarrow{\scriptstyle p} & & \\ \mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}} & & \end{array} \tag{2.17}$$

Lemma 2.1.8. *Notations as in the diagram (2.17) above, then* $M \simeq (\mathscr{O}_{\mathbf{P}}(1) \boxtimes \mathscr{O}_{\mathbf{P}}(2)) \otimes p_* \left(q^* E(-3)\right)$.

Proof. By construction (or see (2.10)),

$$M = p_* \left(q^* E \otimes \mathscr{O}_{\mathbf{P}(S)}(-2D - D')\right), \tag{2.18}$$

where D, D' is the images of the sections σ, σ'. Since p is a projective bundle and the intersection number of D with the fiber is 1, we can assume that $\mathscr{O}_{\mathbf{P}(S)}(-D) = p^* \left(\mathscr{O}_{\mathbf{P}}(a) \boxtimes \mathscr{O}_{\mathbf{P}}(b)\right) \otimes \mathscr{O}_{\mathbf{P}(S)}(-1)$. Pushing forward by p_* the exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}(S)}(-D) \otimes \mathscr{O}_{\mathbf{P}(S)}(1) \to \mathscr{O}_{\mathbf{P}(S)}(1) \to \mathscr{O}_{\mathbf{P}(S)}(1)|_D \to 0,$$

we find an exact sequence:

$$0 \to \mathscr{O}_{\mathbf{P}}(a) \boxtimes \mathscr{O}_{\mathbf{P}}(b) \to S^\vee \to \mathscr{O}_{\mathbf{P}}(1) \boxtimes \mathscr{O}_{\mathbf{P}} \to 0,$$

where the last term comes from the fact that $p_*\left(\mathscr{O}_{\mathbf{P}(S)}(1)|_D\right) = \sigma^*(\mathscr{O}_{\mathbf{P}(S)}(1))$ whose fiber at (y, y') is $(\mathbf{C}\dot{y})^*$. Now noting $S^\vee = \mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \oplus \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(1)$, and restricting to $\mathbf{P}\times\{\mathrm{pt}\}$ and $\{\mathrm{pt}\}\times\mathbf{P}$, we get $a = 0, b = 1$, *i.e.*

$$\mathscr{O}_{\mathbf{P}(S)}(-D) = p^*\left(\mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(1)\right) \otimes \mathscr{O}_{\mathbf{P}(S)}(-1).$$

Similarly, $\mathscr{O}_{\mathbf{P}(S)}(-D') = p^*\left(\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1)\right) \otimes \mathscr{O}_{\mathbf{P}(S)}(-1)$. Putting these into (2.18), the projection formula finishes the proof of Lemma. □

Combining Proposition 2.1.7, Lemma 2.1.5 and Lemma 2.1.8, we have

Proposition 2.1.9. *In* $\mathrm{CH}_n(X^3\backslash\delta_X)$*, we have*

$$\Gamma_o + j_{12*}(Q(h_1,h_2)) + j_{13*}(Q(h_1,h_2)) + j_{23*}(Q(h_1,h_2)) + P(h_1,h_2,h_3) = 0, \tag{2.19}$$

where P is a symmetric homogeneous polynomial of degree $2n$ in three variables with integer coefficients; $h_i \in \mathrm{CH}^1(X^3\backslash\delta_X)$ or $\mathrm{CH}^1(X^2\backslash\Delta_X)$ is the pull-back of $h = c_1(\mathscr{O}_X(1))$ by the i^{th} projection; the inclusions $j_{12}, j_{13}, j_{23} : X^2\backslash\Delta_X \hookrightarrow X^3\backslash\delta_X$ are defined as in Proposition 2.1.7; and Q is a homogeneous polynomial of degree n in two variables with integer coefficients determined by[7]*:*

$$Q(H_1,H_2) = c_{n-r+1}(M)\cdot c_{r-1}\left(\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^*\mathscr{O}_{\mathbf{P}}(1)\otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1)}\right) \in \mathrm{CH}^n(\mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}) \simeq \mathrm{CH}^n(\mathbf{P}^{\times 2}); \tag{2.20}$$

where $M \simeq (\mathscr{O}_{\mathbf{P}}(1)\boxtimes\mathscr{O}_{\mathbf{P}}(2))\otimes p_*(q^*E(-3))$ as in Lemma 2.1.8.

By the localization exact sequence $\mathrm{CH}_n(X) \xrightarrow{\delta_*} \mathrm{CH}_n(X^3) \to \mathrm{CH}_n(X^3\backslash\delta_X) \to 0$, we deduce from (2.19) the following decomposition of small diagonal:

Corollary 2.1.10. *Let P, Q be the same polynomials as in Proposition 2.1.9. Then there exists an integer N, such that we have a decomposition of $N\cdot\delta_X$ in* $\mathrm{CH}_n(X^3)$*:*

$$N\cdot\delta_X = \Gamma + j_{12*}(Q(h_1,h_2)) + j_{13*}(Q(h_1,h_2)) + j_{23*}(Q(h_1,h_2)) + P(h_1,h_2,h_3), \tag{2.21}$$

where we still denote by $j_{12}, j_{23}, j_{13} : X^2 \hookrightarrow X^3$ the inclusions defined by the same formulas as before, and recall that Γ is the subvariety constructed in (2.6), which is the closure of Γ_o in X^3.

In the equation of Corollary 2.1.10 above, we make the following observation of relations between N and the coefficients of P and Q by using the *non-existence* of decomposition of diagonal $\Delta_X \subset X\times X$ in the sense of Bloch-Srinivas (see Theorem 2.0.1 of the introduction) for smooth projective varieties with $H^{n,0}\neq 0$, for example varieties of Calabi-Yau type.

The *degree* of X is given by the maximal self-intersection number of the hyperplane section: $\deg(X) = \Big(\underbrace{h\cdot h\cdots h}_{n}\Big)_X$, which is in fact the top Chern number of E.

Lemma 2.1.11. *Write the* $\mathbf{Z}$*-coefficient polynomials*

$$P(H_1,H_2,H_3) = \sum_{\substack{i+j+k=2n\\ i,j,k\geq 0}} b_{ijk}H_1^iH_2^jH_3^k \quad \textit{with } b_{ijk} \textit{ symmetric on the indexes};$$

$$Q(H_1,H_2) = a_nH_1^n + a_{n-1}H_1^{n-1}H_2 + \cdots + a_0H_2^n.$$

Then we have

$$N = a_0\cdot\deg(X); \tag{2.22}$$

$$a_i + a_j = -b_{ijn}\cdot\deg(X) \quad \textit{for any } i+j=n. \tag{2.23}$$

7. Here $\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^*\mathscr{O}_{\mathbf{P}}(1)\otimes\mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1)}$ is no more a quotient vector bundle, but only an element in the Grothendieck group of vector bundles on $\mathbf{P}^{\times 2}$ on which the Chern classes are however still well-defined.

Proof. Applying to the equation in Corollary 2.1.10 the push-forward induced by the projection to the first two factors $\mathrm{pr}_{12} : X^3 \twoheadrightarrow X^2$, then

- $\mathrm{pr}_{12*}(N \cdot \delta_X) = N \cdot \Delta_X$;
- $\mathrm{pr}_{12*}(\Gamma) = 0$, since Γ has relative dimension 1 for pr_{12};
- $\mathrm{pr}_{12*} \circ j_{12}\,(Q(h_1, h_2)) = \Delta_* \circ \mathrm{pr}_{1*}\,(Q(h_1, h_2)) = a_0 \deg(X) \cdot \Delta_X$;
- $\mathrm{pr}_{12*} \circ j_{13}\,(Q(h_1, h_2)) = \mathrm{id}_{X^2*}\,(Q(h_1, h_2)) = Q(h_1, h_2)$;
- $\mathrm{pr}_{12*} \circ j_{23}\,(Q(h_1, h_2)) = \iota_{X^2*}\,(Q(h_1, h_2)) = Q(h_2, h_1)$, where $\iota : X^2 \to X^2$ is the the involution interchanging the two coordinates;
- $\mathrm{pr}_{12*}\,(P(h_1, h_2, h_3)) = \deg(X) \cdot \sum_{i+j=n} b_{ijn} h_1^i h_2^j$.

Putting these together, we obtain that in $\mathrm{CH}_n(X^2)$,

$$(N - a_0 \cdot \deg(X)) \cdot \Delta_X = \sum_{i+j=n} (b_{ijn} \deg(X) + a_i + a_j) \cdot h_1^i h_2^j. \tag{2.24}$$

If $N - a_0 \cdot \deg(X) \neq 0$, then (2.24) gives a nontrivial decomposition of diagonal $\Delta_X \in \mathrm{CH}_n(X^2)$ of Bloch-Srinivas type, but this is impossible: we regard (2.24) as an equality of cohomological correspondences from $H^n(X)$ to itself, then the left hand side acts by multiplying a non-zero constant $(N - a_0 \cdot \deg(X))$, while the action of the right hand side has image a sub-Hodge structure of coniveau at least 1, which contradicts to the non-vanishing of $H^{n,0}(X)$.
As a result, we have (2.22), and hence (2.23) since $\left\{h_1^i h_2^j\right\}_{i+j=n}$ are linearly independent. □

Using Lemma 2.1.11, we obtain improved version of Corollary 2.1.10 as summarized in the following

Theorem 2.1.12. *Let* $\mathbf{P}, E, X$ *be as in the basic setting. Then there is a homogeneous polynomial with* $\mathbf{Z}$*-coefficients*

$$Q(H_1, H_2) = a_n H_1^n + a_{n-1} H_1^{n-1} H_2 + \cdots + a_0 H_2^n,$$

which is determined by (2.20), such that in $\mathrm{CH}_n(X^3)$

$$a_0 \deg(X) \cdot \delta_X = \Gamma + j_{12*}(Q(h_1, h_2)) + j_{13*}(Q(h_1, h_2)) + j_{23*}(Q(h_1, h_2)) + P(h_1, h_2, h_3), \tag{2.25}$$

where P *is the symmetric polynomial with* $\mathbf{Z}$*-coefficients*

$$P(H_1, H_2, H_3) = \sum_{i+j+k=2n} b_{ijk} H_1^i H_2^j H_3^k,$$

with (2.23): $a_i + a_j = -b_{ijn} \cdot \deg(X)$ *for any* $i + j = n$*; and* $h_i \in \mathrm{CH}^1(X^3)$ *or* $\mathrm{CH}^1(X^2)$ *is the pull-back of* $h = c_1(\mathscr{O}_X(1))$ *by the* i^{th} *projection, and the inclusions of big diagonals are given by:*

$$\begin{array}{rll} & X^2 & \hookrightarrow X^3 \\ j_{12}: & (x, x') & \mapsto (x, x, x') \\ j_{13}: & (x, x') & \mapsto (x, x', x) \\ j_{23}: & (x, x') & \mapsto (x', x, x). \end{array}$$

This theorem generalizes the result of Voisin [75] for Calabi-Yau hypersurfaces (see Theorem 2.0.5 of the introduction) except for a small point: to get a non-trivial decomposition of the small diagonal and thus applications like Corollary 2.0.6, we need to verify that $a_0 \neq 0$. It is the case when E is splitting, *i.e.* when X is Calabi-Yau complete intersection, see the following subsections.

2.1.2 Applications to the multiplicative structure of Chow rings

Always in the same setting as the previous subsection. Now we can regard (2.25) as an equality of correspondences from $X \times X$ to X. Combining with (2.23), we get the following corollary in the same spirit of Corollary 2.0.6 of the introduction.

Corollary 2.1.13. *In the same notation as before. Let* $Z \in \mathrm{CH}^k(X), Z' \in \mathrm{CH}^l(X)$ *be two algebraic cycles of* X *of codimension* $k, l \in \mathbf{N}$ *with* $k + l = n$*. Then we have an equality in* $\mathrm{CH}_0(X)$*, here* $\bullet$ *means the intersection product in* $\mathrm{CH}^*(X)$*:*

$$a_0 \deg(X)\cdot Z\bullet Z' = a_0 \deg(Z\bullet Z')\cdot h^n + a_l \deg(Z')\cdot Z\bullet h^l + a_k \deg(Z)\cdot Z'\bullet h^k - (a_k+a_l)\frac{\deg(Z)\deg(Z')}{\deg(X)}\cdot h^n,$$

where the degree *of an algebraic cycle* $\mathscr{Z}$ *is defined to be the intersection number* $(\mathscr{Z}\cdot h^{\dim \mathscr{Z}})_X$.

Proof. Let us list term by term the results of applying the correspondences in (2.25) to the cycle $Z \times Z' \in \mathrm{CH}^n(X\times X)$:

- $(a_0 \deg(X)\cdot \delta_X)_* (Z\times Z') = a_0 \deg(X)\cdot Z \bullet Z'$;
- $\Gamma_*(Z\times Z') = 0$, since $\Gamma_*(Z\times Z')$ is represented by a linear combination of some fundamental class of a subvariety of dimension at least 1, but $\Gamma_*(Z\times Z')$ should be a zero-dimensional cycle, so it vanishes;
- $\left(j_{12*}(h_1^i h_2^{n-i})\right)_* (Z\times Z') = (Z\cdot Z'\cdot h^i)_X \cdot h^n$ if $i = 0$, and vanishes otherwise, therefore
$$\left(j_{12*}Q(h_1,h_2)\right)_* (Z\times Z') = a_0 \deg(Z\bullet Z')\cdot h^n;$$
- $\left(j_{13*}(h_1^i h_2^{n-i})\right)_* (Z\times Z') = (Z'\cdot h^{n-l})_X \cdot Z\bullet h^l$ if $i = l$, and vanishes otherwise, therefore
$$\left(j_{13*}Q(h_1,h_2)\right)_* (Z\times Z') = a_l \deg(Z')\cdot Z\bullet h^l;$$
- $\left(j_{23*}(h_1^i h_2^{n-i})\right)_* (Z\times Z') = (Z\cdot h^{n-k})_X \cdot Z'\bullet h^k$ if $i = k$, and vanishes otherwise, therefore
$$\left(j_{23*}Q(h_1,h_2)\right)_* (Z\times Z') = a_k \deg(Z)\cdot Z'\bullet h^k;$$
- $(h_1^{i_1}h_2^{i_2}h_3^{i_3})_*(Z\times Z') = (h^l\cdot Z)_X\cdot(h^k\cdot Z')_X\cdot h^n$ if $(i_1,i_2,i_3) = (l,k,n)$, and vanishes otherwise, therefore
$$\left(P(h_1,h_2,h_3)\right)_* (Z\times Z') = b_{l,k,n}\deg(Z)\deg(Z')\cdot h^n.$$

Putting all these together, we deduce exactly what we want. □

To simplify further the equality in Corollary 2.1.13, we need:

Lemma 2.1.14. *Suppose* $a_1 \neq a_0$*. Let* $k \in \{0,1,\cdots,n-1\}$*. Then for any* $\mathscr{Z} \in \mathrm{CH}^k(X)_{\mathbf{Q}}$*, we have* $\mathscr{Z}\bullet h^{n-k}$ *is always proportional to* h^n *in* $\mathrm{CH}_0(X)_{\mathbf{Q}}$.

Proof. Given $\mathscr{Z} \in \mathrm{CH}^k(X)_{\mathbf{Q}}$, replacing Z by $\mathscr{Z}\bullet h^{n-k-1}$ and Z' by h in the formula of Corollary 2.1.13, we get:

$$(a_0 - a_1)\deg(X)\cdot\left(Z\bullet h^{n-k} - \frac{\deg(\mathscr{Z})}{\deg(X)}h^n\right) = 0.$$

Since $a_0 \neq a_1$, we can divide out $(a_0 - a_1)\deg(X)$, obtaining $Z\bullet h^{n-k} = \frac{\deg(Z)}{\deg(X)}h^n$. □

Inserting this lemma into the formula of Corollary 2.1.13, its last three terms simplify with each other, and we finally obtain our main consequence of the decomposition theorem:

Theorem 2.1.15. *Let E be a rank r vector bundle on $\mathbf{P}^{n+r}$ satisfying the positivity condition $(*)$ as well as the Calabi-Yau condition:* $\det(E) \simeq \mathscr{O}_{\mathbf{P}}(n+r+1)$. *Let a_i still be the coefficients determined by (2.20). Suppose $a_0 \neq 0$ and $a_1 \neq a_0$. Let X be the (Calabi-Yau) zero locus of a general section of E. Then for any strictly positive integers $k, l \in \mathbf{N}^*$, with $k + l = n$.*

$$\mathrm{Im}\left(\bullet : \mathrm{CH}^k(X)_{\mathbf{Q}} \times \mathrm{CH}^l(X)_{\mathbf{Q}} \to \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n,$$

where $h = c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X)$.

2.1.3 Splitting case: Calabi-Yau complete intersections

In this subsection, we deal with the special case that E is of splitting type:

$$E = \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbf{P}}(d_i),$$

where $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$, with $d = \sum_{i=1}^r d_i = n + r + 1$. Hence X is a smooth Calabi-Yau complete intersection of multi-degree $(d_1, \cdots, d_r)$. In particular, $\deg(X) = \prod_{i=1}^r d_i$.

We are going to prove that the conditions $a_0 \neq 0$ and $a_1 \neq a_0$ appeared in Theorem 2.1.15 are satisfied in this special case by the following calculations:

Lemma 2.1.16. *If E is of splitting type as above, then*

1. *The vector bundle M in Lemma 2.1.8 is isomorphic to the restriction to $\mathbf{P}^{\times 2} \backslash \Delta_{\mathbf{P}}$ of*

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{d_i - 2} \mathscr{O}_{\mathbf{P}}(j) \boxtimes \mathscr{O}_{\mathbf{P}}(d_i - j).$$

   In particular in (2.20),

$$c_{n-r+1}(M) = \prod_{i=1}^{r} \prod_{j=1}^{d_i-2} (jH_1 + (d_i - j)H_2). \tag{2.26}$$

2. *In (2.20), $c_{r-1}\left(\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)}\right)$ is the degree $(r-1)$ part of the formal series*

$$\frac{\prod_{i=1}^r (1 + d_i H_1)}{1 - (H_2 - H_1)}.$$

3. *The coefficient of H_2^n in the polynomial Q is given by $a_0 = \prod_{i=1}^r ((d_i - 1)!)$, which is non-zero in particular.*

4. *The coefficient of $H_1 H_2^{n-1}$ in the polynomial Q is given by*

$$a_1 = \left(\prod_{i=1}^{r} (d_i - 1)!\right) \cdot \left(\left(\sum_{i=1}^{r} \sum_{j=1}^{d_i-2} \frac{j}{d_i - j}\right) + n + 2\right),$$

   in particular $a_1 \neq a_0$.

Proof. In the situation as in diagram (2.17),

$$\begin{aligned}
p_*\left(q^*E(-3)\right) &= \oplus_{i=1}^r p_*q^*\mathscr{O}_{\mathbf{P}}(d_i-3) && \\
&= \oplus_{i=1}^r p_*\mathscr{O}_{\mathbf{P}(S)}(d_i-3) && (\text{since } q^*\mathscr{O}_{\mathbf{P}}(1)=\mathscr{O}_{\mathbf{P}(S)}(1)) \\
&= \oplus_{i=1}^r \operatorname{Sym}^{d_i-3} S^\vee && (d_i-3\geq -1, \text{define } \operatorname{Sym}^0=\mathscr{O}, \operatorname{Sym}^{-1}=0) \\
&= \oplus_{i=1}^r \oplus_{j=0}^{d_i-3}\left(\mathscr{O}_{\mathbf{P}}(j)\boxtimes\mathscr{O}_{\mathbf{P}}(d_i-3-j)\right) && (\text{recall } S\simeq \operatorname{pr}_1^*\mathscr{O}_{\mathbf{P}}(-1)\oplus\operatorname{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1))
\end{aligned}$$

Thus $M\simeq(\mathscr{O}_{\mathbf{P}}(1)\boxtimes\mathscr{O}_{\mathbf{P}}(2))\otimes p_*\left(q^*E(-3)\right)=\bigoplus_{i=1}^r\bigoplus_{j=1}^{d_i-2}\left(\mathscr{O}_{\mathbf{P}}(j)\boxtimes\mathscr{O}_{\mathbf{P}}(d_i-j)\right)$, and the top Chern class follows immediately.
The second point is obvious. As for the coefficient a_0, by (2.20) it is the product of the coefficient of H_2^{n-r+1} in $c_{n-r+1}(M)$ and the coefficient of H_2^{r-1} in $c_{r-1}\left(\frac{\operatorname{pr}_1^*E}{\operatorname{pr}_1^*\mathscr{O}_{\mathbf{P}}(1)\otimes\operatorname{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1)}\right)$, which are $\prod_{i=1}^r(d_i-1)!$ and 1 respectively, by the first two parts of this lemma. Remembering $\left(\sum_{i=1}^r d_i\right)-r+1=n+2$, the calculation for a_1 is also straightforward. □

As a result, in this complete intersection case the decomposition Theorem 2.1.12 and its application Theorem 2.1.15 on the multiplicative structure on the Chow rings then read as following respectively:

Theorem 2.1.17. *Let X be a general Calabi-Yau complete intersection in a projective space, then in* $\mathrm{CH}_n(X^3)$ *we have a decomposition of the small diagonal:*

$$\left(\prod_{i=1}^r(d_i!)\right)\cdot\delta_X=\Gamma+j_{12*}(Q(h_1,h_2))+j_{13*}(Q(h_1,h_2))+j_{23*}(Q(h_1,h_2))+P(h_1,h_2,h_3),$$

Theorem 2.1.18. *Let X be a general Calabi-Yau complete intersection in a projective space, then for any strictly positive integers* $k,l\in\mathbf{N}^*$, *with* $k+l=n$.

$$\operatorname{Im}\left(\bullet:\mathrm{CH}^k(X)_{\mathbf{Q}}\times\mathrm{CH}^l(X)_{\mathbf{Q}}\to\mathrm{CH}_0(X)_{\mathbf{Q}}\right)=\mathbf{Q}\cdot h^n,$$

where $h=c_1(\mathscr{O}_X(1))\in\mathrm{CH}^1(X)$.

Remark 2.1.19. Theorem 2.1.18 can be reformulated as: for a general Calabi-Yau complete intersection in a projective space, any *decomposable* 0-cycle with **Q**-coefficient is **Q**-rational equivalent to zero if and only if it has degree 0. Here a 0-cycle is called *decomposable* if it is in the sum of the images:

$$\sum_{\substack{k+l=n\\k,l>0}}\operatorname{Im}\left(\mathrm{CH}^k(X)_{\mathbf{Q}}\times\mathrm{CH}^l(X)_{\mathbf{Q}}\xrightarrow{\bullet}\mathrm{CH}_0(X)_{\mathbf{Q}}\right).$$

It is interesting to compare this result for Calabi-Yau varieties with Beauville's *'weak splitting principle'* conjecture in the holomorphic symplectic case. In [12], Beauville and Voisin reinterpreted their result (see Corollary 2.0.4 in §0 Introduction) as some sort of compatibility with splitting of the conjectural Bloch-Beilinson filtration. In [10], Beauville proposed (and checked some examples of) a weak form of such compatibility for higher dimensional irreducible holomorphic symplectic varieties to check, namely his 'weak splitting property' conjecture. Later in [71], Voisin formulates a stronger version of this conjecture: *for an irreducible holomorphic symplectic projective variety, any polynomial relation between the* cohomological *Chern classes of lines bundles and the tangent bundle holds already for their* Chow-theoretical *Chern classes*. She also proved this conjecture for the variety of lines in a cubic four-fold and for Hilbert schemes of points on K3 surfaces in certain range (*cf.* [71]).

Note that the Chern classes of the tangent bundle of a complete intersection is given by

$$c(T_X) = \left(\frac{c(T_{\mathbf{P}})}{c(E)}\right)\Big|_X,$$

which is clearly a cycle coming from the ambient projective space. Therefore comparing to the 'weak splitting principle' for holomorphic symplectic varieties, our result for Calabi-Yau varieties is on the one hand stronger, in the sense that besides the divisors and Chern classes of the tangent bundle, cycles of all strictly positive codimension are allowed in the polynomial; and on the other hand weaker since only the polynomials of (weighted) degree n are taken into account.

Remark 2.1.20 (non-splitting case)**.** In this remark, we discuss the possibility to generalize the above results to the broadest case of X being the zero locus of a general section of vector bundle satisfying the positivity condition $(*)$ and Calabi-Yau condition $\det(E) \simeq \mathscr{O}_{\mathbf{P}}(n+r+1)$. By Theorem 2.1.15, to draw the same conclusion that any degree 0 decomposable $\mathbf{Q}$-0-cycle is rationally equivalent to 0 (*cf.* Remark 2.1.19), the only thing we need is to verify that the coefficients of the polynomial Q defined in (2.20) satisfy the conditions
$(\diamond\diamond)$: $a_0 \neq 0$ and $a_1 \neq a_0$.
Although no proof is found for the time being, the author feels that very probably $(\diamond\diamond)$ is always true under the condition $(*)$. In fact, once the Chern classes of E are given, the coefficients a_0 and a_1 are practically computable: Grothendieck-Riemann-Roch formula provides us the Chern character of the vector bundle M in Lemma 2.1.8:

$$\operatorname{ch}(M)(t) = e^{(H_1+2H_2)t} \cdot \sum_{i=1}^{r} \frac{e^{(d_i-2)H_1 t} - e^{(d_i-2)H_2 t}}{e^{H_1 t} - e^{H_2 t}}, \tag{2.27}$$

from which $c_{n-r+1}(M)$ and thus

$$Q(H_1, H_2) = c_{n-r+1}(M) \cdot c_{r-1}\left(\frac{\operatorname{pr}_1^* E}{\operatorname{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \operatorname{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)}\right)$$

could be calculated without essential difficulties in practice. It is clear at this point that, the condition $(\diamond\diamond)$ is an 'open' condition; while the calculation for the splitting case (see the proof of Lemma 2.1.16) seems to imply that the positivity condition $(*)$ makes $(\diamond\diamond)$ far from being wrong. It is possible that some theory for the classification of (Chern classes of) vector bundles would be needed to get a genuine proof. A detailed discussion is given in the next subsection.

2.1.4 General case

After working out the complete intersection case above, let us now return to the general case, namely when $X = (f = 0)$ is the zero locus of a section f of a rank r vector bundle E on $\mathbf{P} := \mathbf{P}^{n+r}$ satisfying the positivity condition $(*)$, such that X is smooth of Calabi-Yau type: $d = c_1(E) = n + r + 1$.

Firstly, we observe:

Remark 2.1.21. After a careful examination of the proof of Theorem 2.1.18 from Corollary 2.1.13, we find that the only place we used the complete intersection assumption is that $a_0 \neq 0$ and $a_i \neq a_0$ for any $i \geq 1$. In other words: if the coefficients of the polynomial Q defined in (2.20) satisfy $a_0 \neq 0$ and $a_i \neq a_0$, then for any strictly positive integers $k, l \in \mathbf{N}^*$ with $k + l = n$, we have

$$\operatorname{Im}\left(\bullet : \operatorname{CH}^k(X)_{\mathbf{Q}} \times \operatorname{CH}^l(X)_{\mathbf{Q}} \to \operatorname{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n,$$

where $h = c_1(\mathscr{O}_X(1)) \in \operatorname{CH}^1(X)$.

The rest of this subsection will be devoted to express the condition $a_0 \neq 0$ and $a_i \neq a_0$ in terms of Chern classes of E. Suppose that:

$$c(E) = \prod_{i=1}^{r} (1 + d_i H) \ \text{ in } \ \mathrm{CH}^*(\mathbf{P}), \tag{2.28}$$

where $d_i \in \overline{\mathbf{Q}}$ are just algebraic integers in general, called the *Chern roots* of E. The Calabi-Yau condition is simply $d := \sum_{i=1}^{r} d_i = n + r + 1$.

Recall the definition (2.20) of Q: in $\mathrm{CH}^n(\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}) \simeq \mathrm{CH}^n(\mathbf{P}^{\times 2})$,

$$Q(H_1, H_2) = c_{n-r+1}(M) \cdot c_{r-1}\left(\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)}\right),$$

where $M = (\mathscr{O}_{\mathbf{P}}(1) \boxtimes \mathscr{O}_{\mathbf{P}}(2)) \otimes p_* \left(q^* E(-3)\right)$, where p, q are as in the following diagram:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{\ q\ } & \mathbf{P} \\ \downarrow{\scriptstyle p} & & \\ \mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}} & & \end{array} \tag{2.29}$$

Here S is the tautological rank 2 bundle.

The main difficulty is that unlike the splitting case, $p_*(q^* E(-3))$ is hard to identify in general. Our approach to its top Chern class will be through the Grothendieck-Riemann-Roch formula, however the passage from the Chern characters to the top Chern class will cause some complication in the calculation.

Lemma 2.1.22. *We have*

$$\begin{aligned} c_{r-1}\left(\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)}\right) &= \sum_{j=0}^{r-1} \sum_{i=0}^{j} (-1)^{i+j} c_i(E) \binom{r-1-i}{r-1-j} H_1^j H_2^{r-1-j} \\ &= H_2^{r-1} + (n+2)\, H_1 H_2^{r-2} + \cdots . \end{aligned}$$

Proof. It is straightforward:

$$\begin{aligned} c\left(\frac{\mathrm{pr}_1^* E}{\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)}\right) &= \frac{c\left(\mathrm{pr}_1^* E\right)}{c\left(\mathrm{pr}_1^* \mathscr{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^* \mathscr{O}_{\mathbf{P}}(-1)\right)} \\ &= \frac{1 + c_1(E) H_1 + c_2(E) H_1^2 + \cdots + c_r(E) H_1^r}{1 + H_1 - H_2} \end{aligned}$$

Therefore the coefficient of H_2^{r-1} is obviously 1, and the coefficient of $H_1 H_2^{r-2}$ is

$$-(r-1) + c_1(E) = d - r + 1 = n + 2.$$

□

Before the calculation of the Chern character of $M = (\mathscr{O}_{\mathbf{P}}(1) \boxtimes \mathscr{O}_{\mathbf{P}}(2)) \otimes p_* \left(q^* E(-3)\right)$, we state an elementary lemma:

Lemma 2.1.23. *Let R be a commutative ring which is finitely generated over $\mathbf{Z}$, such that every generator is nilpotent. Let $f(T) \in R[[T]]$ be a power series with coefficient in the ring R, and $b, c \in R$ be nilpotent elements. Suppose $f(T) \equiv \lambda T + \mu \pmod{(T-b)^2 - c^2}$ with $\lambda, \mu \in R$. Then $\lambda = \frac{f(b+c) - f(b-c)}{2c}$.*

Proof. Write $f(T) = a_0 + a_1(T-b) + a_2(T-b)^2 + a_3(T-b)^3 \cdots$, then

$$f(b+c) = a_0 + a_1 c + a_2 c^2 + a_3 c^3 + \cdots;$$

$$f(b-c) = a_0 - a_1 c + a_2 c^2 - a_3 c^3 + \cdots,$$

thus $\frac{f(b+c)-f(b-c)}{2c} = a_1 + a_3 c^2 + a_5 c^4 \cdots$. However, modulo $(T-b)^2 = c^2$,

$$f(T) \equiv a_0 + a_1(T-b) + a_2 c^2 + a_3 c^2 (T-b) + \cdots,$$

therefore $\lambda = a_1 + a_3 c^2 + a_5 c^4 + \cdots$, which proves the lemma. □

Proposition 2.1.24. *The Chern character of* $M = (\mathscr{O}_{\mathbf{P}}(1) \boxtimes \mathscr{O}_{\mathbf{P}}(2)) \otimes p_*(q^* E(-3))$ *is given by the following formula:*

$$\mathrm{ch}(M)(t) = e^{(H_1+2H_2)t} \cdot \sum_{i=1}^{r} \frac{e^{(d_i-2)H_1 t} - e^{(d_i-2)H_2 t}}{e^{H_1 t} - e^{H_2 t}}, \tag{2.30}$$

where the variable t is introduced to keep track of the degree.

Proof. We are in the situation of the diagram (2.29). Define $\xi := c_1(\mathscr{O}_{\mathbf{P}(S)}(1)) \in \mathrm{CH}^1(\mathbf{P}(S))$, and let us denote still by H_1, H_2 their pull-backs by p^*, thus we have:

$$\xi^2 - (H_1 + H_2)\xi + H_1 H_2 = 0.$$

Thanks to the positivity assumption $(*'')$, we have $R^1 p_*(q^*(E(-3))) = 0$. Thus by the Grothendieck-Riemann-Roch theorem:

$$\begin{aligned} \mathrm{ch}\left(p_*\left(q^* E(-3)\right)\right) &= p_*\left(\mathrm{ch}\left(q^* E(-3)\right) \cdot \mathrm{td}\left(T_{\mathbf{P}(S)/\mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}}}\right)\right) \\ &= p_*\left(q^* \,\mathrm{ch}\left(E(-3)\right) \cdot \mathrm{td}\left(p^* S \otimes \mathscr{O}_{\mathbf{P}(S)}(1)\right)\right) \\ &= p_*\left(\left(\sum_{i=1}^{r} e^{(d_i-3)\xi}\right) \cdot \frac{\xi - H_1}{1 - e^{H_1-\xi}} \cdot \frac{\xi - H_2}{1 - e^{H_2-\xi}}\right). \end{aligned}$$

Note that in general $p_*(\lambda\xi + \mu) = \lambda$ for $\lambda, \mu \in \mathrm{CH}^*(\mathbf{P}^{\times 2}\backslash\Delta_{\mathbf{P}})$. Since $H_1^{n+r+1} = H_2^{n+r+1} = 0$, and $\left(\xi - \frac{H_1+H_2}{2}\right)^2 = \left(\frac{H_1-H_2}{2}\right)^2$, we can apply the preceding lemma to obtain:

$$\begin{aligned} \mathrm{ch}\left(p_*\left(q^* E(-3)\right)\right) &= \frac{\left(\sum_{i=1}^r e^{(d_i-3)H_1}\right) \cdot \frac{H_1-H_2}{1-e^{H_2-H_1}} - \left(\sum_{i=1}^r e^{(d_i-3)H_2}\right) \cdot \frac{H_2-H_1}{1-e^{H_1-H_2}}}{H_1 - H_2} \\ &= \frac{\sum_i \left(e^{(d_i-2)H_1} - e^{(d_i-2)H_2}\right)}{e^{H_1} - e^{H_2}}. \end{aligned}$$

The formula in the lemma then follows by the multiplicativity of Chern characters. □

Remark 2.1.25. In the splitting case of the preceding subsection, *i.e.* the d_i's are integers at least 2, then the above lemma gives immediately:

$$\mathrm{ch}(M)(t) = \sum_{i=1}^{r} \left(e^{(d_i-2)H_1+2H_2} + e^{(d_i-3)H_1+3H_2} + \cdots + e^{H_1+(d_i-1)H_2}\right),$$

and hence the Chern roots $\{jH_1 + (d_i - j)H_2\}_{1\le i\le r, 1\le j\le d_i-2}$, from which we can read off the top chern class of M easily, recovering the calculation (2.26).

Back to the general case, we want to pass from Chern characters calculated in the preceding lemma to its top Chern class. we denote the above power series by

$$\varphi(t) := \mathrm{ch}(M)(t) = e^{(H_1+2H_2)t} \cdot \sum_{i=1}^{r} \frac{e^{(d_i-2)H_1 t} - e^{(d_i-2)H_2 t}}{e^{H_1 t} - e^{H_2 t}}. \tag{2.31}$$

Then in its Taylor expansion

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2!}\varphi''(0)t^2 + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + \cdots,$$

the k-th derivative $\varphi^{(k)}(0)$ is a integral coefficient homogeneous polynomial in H_1, H_2 of degree k, whose coefficients are related to the Chern classes of E. For example:

$$\varphi(0) = \mathrm{rank}(M) = \sum_{i=1}^{r} d_i - 2r = n - r + 1;$$

$$\varphi'(0) = \mathrm{ch}_1(M) = \left(\frac{1}{2}c_1(E)^2 - \frac{3}{2}c_1(E) - c_2(E) + r\right)H_1 + \left(\frac{1}{2}c_1(E)^2 - \frac{1}{2}c_1(E) - c_2(E) - r\right)H_2;$$
$$\cdots$$

By the theory of symmetric polynomials, or more precisely a consequence of Newton's identities, we have the following determinantal formula:

$$c_{n-r+1}(M) = \frac{1}{(n-r+1)!}\begin{vmatrix} \varphi'(0) & 1 & 0 & \cdots & & \\ \varphi''(0) & \varphi'(0) & 2 & 0 & \cdots & \\ \vdots & & \ddots & \ddots & & \\ \varphi^{(n-r)}(0) & \varphi^{(n-r-1)}(0) & \cdots & \varphi'(0) & n-r \\ \varphi^{(n-r+1)}(0) & \varphi^{(n-r)}(0) & \cdots & \varphi''(0) & \varphi'(0) \end{vmatrix}. \tag{2.32}$$

Therefore we need to calculate the derivatives of φ. This is treated in the following lemma:

Lemma 2.1.26. *The derivatives $\varphi^{(k)}(0)$ are determined by:*

$$\begin{pmatrix} \varphi(0) \\ \varphi'(0) \\ \varphi''(0) \\ \varphi^{(3)}(0) \\ \varphi^{(4)}(0) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ T_1 & 2 & 0 & 0 & 0 & \cdots \\ T_2 & 3T_1 & 3 & 0 & 0 & \cdots \\ T_3 & 4T_2 & 6T_1 & 4 & 0 & \cdots \\ T_4 & 5T_3 & 10T_2 & 10T_1 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2J & 1 & 0 & 0 & 0 & \cdots \\ 3J^2 & 3J & 1 & 0 & 0 & \cdots \\ 4J^3 & 6J^2 & 4J & 1 & 0 & \cdots \\ 5J^4 & 10J^3 & 10J^2 & 5J & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} s_1 \\ s_2T_1 \\ s_3T_2 \\ s_4T_3 \\ s_5T_4 \\ \vdots \end{pmatrix}$$

where $\varphi(0) = n - r + 1$, the integer $s_j = \sum_{i=1}^{r}(d_i - 2)^j = j! \cdot \mathrm{ch}_j(E(-2))$, and the polynomial $J(H_1, H_2) = H_1 + 2H_2$ and $T_j(H_1, H_2) = H_1^j + H_1^{j-1}H_2 + \cdots + H_2^j$

Proof. We get from (2.31) that:

$$\left(e^{H_1 t} - e^{H_2 t}\right)\varphi(t) = e^{(H_1+2H_2)t} \cdot \sum_{i=1}^{r}\left(e^{(d_i-2)H_1 t} - e^{(d_i-2)H_2 t}\right).$$

Taking the k-th derivatives on both sides, evaluating at $t = 0$, and simplifying the common factor $(H_1 - H_2)$, we obtain that for any $k \geq 1$,

$$\binom{k}{1}\varphi^{(k-1)}(0) + \binom{k}{2}\varphi^{(k-2)}(0) \cdot T_1 + \binom{k}{3}\varphi^{(k-3)}(0) \cdot T_2 + \cdots + \binom{k}{k-1}\varphi'(0) \cdot T_{k-2} + \varphi(0) \cdot T_{k-1}$$
$$= \binom{k}{1}s_1 J^{k-1} + \binom{k}{2}s_2 \cdot J^{k-2}T_1 + \cdots + \binom{k}{k-1}s_{k-1} \cdot JT_{k-2} + s_k \cdot T_{k-1}.$$

where the integers s_j and polynomials$J$ and T_j are defined in the statement of the lemma. Rewriting these using matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ T_1 & 2 & 0 & 0 & \cdots \\ T_2 & 3T_1 & 3 & 0 & \cdots \\ T_3 & 4T_2 & 6T_1 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi'(0) \\ \varphi''(0) \\ \varphi'''(0) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2J & 1 & 0 & 0 & \cdots \\ 3J^2 & 3J & 1 & 0 & \cdots \\ 4J^3 & 6J^2 & 4J & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} s_1 \\ s_2T_1 \\ s_3T_2 \\ s_4T_3 \\ \vdots \end{pmatrix}$$

Inverting the lower-triangular matrix on the left hand side finishes the proof. □

Combining all the calculations (2.32), Lemma 2.1.26 and Lemma 2.1.22, we can calculate in practice all the coefficients of the polynomial Q, but I do not think there is a way to write a compact formula to express them.

For example, for a_0 and a_1:

Proposition 2.1.27. *For any positive integer j, let s_j be the integer such that $j!\,\mathrm{ch}_j(E) = s_j \cdot H^j$ in $\mathrm{CH}^j(\mathbf{P})$. Introduce a formal variable ϵ with $\epsilon^2 = 0$. For any $j \in \mathbf{N}$, define $\phi_j \in \mathbf{Z}[\epsilon]/(\epsilon^2)$ by: $\phi_0 = n - r + 1$ and*

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1+\epsilon & 2 & 0 & 0 & \cdots \\ 1+\epsilon & 3(1+\epsilon) & 3 & 0 & \cdots \\ 1+\epsilon & 4(1+\epsilon) & 6(1+\epsilon) & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2(2+\epsilon) & 1 & 0 & 0 & \cdots \\ 3(2+\epsilon)^2 & 3(2+\epsilon) & 1 & 0 & \cdots \\ 4(2+\epsilon)^3 & 6(2+\epsilon)^2 & 4(2+\epsilon) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} s_1 \\ s_2(1+\epsilon) \\ s_3(1+\epsilon) \\ s_4(1+\epsilon) \\ \vdots \end{pmatrix}$$

Let λ, μ be such that

$$\lambda + \mu\epsilon = \frac{1}{(n-r+1)!} \begin{vmatrix} \phi_1 & 1 & 0 & \cdots & \\ \phi_2 & \phi_1 & 2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ \phi_{n-r} & \phi_{n-r-1} & \cdots & \phi_1 & n-r \\ \phi_{n-r+1} & \phi_{n-r} & \cdots & \phi_2 & \phi_1 \end{vmatrix}.$$

Then $a_0 = \lambda$ and $a_1 = \mu + (n+2)\lambda$.

Proof. Since our interest is only on the H_2^n term and $H_2^{n-1}H_1$ term, and everything involved is homogeneous in H_1 and H_2, we can thus do the calculation by regarding H_1 as a 'small correction term'. Now it is an immediate consequence of Lemma 2.1.22, formula (2.32), Lemma 2.1.26. □

Based on the explicite calculation of coefficients a_i of Q above, we introduce two numerical conditions on the Chern classes of E.

Definition 2.1.28. Let E be a rank r vector bundle on the projective space $\mathbf{P}^{n+r}$ with Chern characters $\mathrm{ch}_j(E) = \frac{1}{j!} s_j \cdot H^j \in \mathrm{CH}^j(\mathbf{P}^{n+r})_{\mathbf{Q}}$.

1. We say E satisfies the condition $(\lozenge)$, if

$$\begin{vmatrix} \psi_1 & 1 & 0 & \cdots & \\ \psi_2 & \psi_1 & 2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ \psi_{n-r} & \psi_{n-r-1} & \cdots & \psi_1 & n-r \\ \psi_{n-r+1} & \psi_{n-r} & \cdots & \psi_2 & \psi_1 \end{vmatrix} \neq 0,$$

where

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 0 & \cdots \\ 1 & 4 & 6 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2\cdot 2 & 1 & 0 & 0 & \cdots \\ 3\cdot 2^2 & 3\cdot 2 & 1 & 0 & \cdots \\ 4\cdot 2^3 & 6\cdot 2^2 & 4\cdot 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \end{pmatrix}$$

with $\psi_0 = n - r + 1$, the numbers appeared above are binomial numbers.

2. Similarly, we say E satisfies condition ($\Diamond\Diamond$), if moreover, the a_i's determined by the calculations (2.32), Lemma 2.1.26 and Lemma 2.1.22 satisfy $a_0 \neq a_i$ for all $i \geq 1$. For example, using the notation in Proposition 2.1.27, we have $a_1 \neq a_0$ if and only if $\mu + (n+1)\lambda \neq 0$.

Remark 2.1.29. As explained in the last subsection and Remark 2.1.25, the splitting vector bundles with our positivity condition ($*$) automatically satisfy the condition ($\Diamond\Diamond$). In general, despite of their unpleasant appearances, the conditions ($\Diamond$) and ($\Diamond\Diamond$) seem to be 'closed' conditions, and the author believes that this condition is in fact automatically satisfied in the presence of our positivity condition ($*$), but maybe its proof needs some further results on the classification of Chern classes for vector bundles on projective spaces. Anyway, it seems to be at least a fairly weak condition which should not cause any problem in practice.

Then Theorem 2.1.12 is improved into the part (i) of Theorem 2.0.7 in the introduction.

Finally we can restate the observation in Remark 2.1.21 as a theorem:

Theorem 2.1.30. *Let E be a rank r vector bundle on $\mathbf{P}^{n+r}$ satisfying the positivity condition ($*$), the Calabi-Yau condition $c_1(E) = n + r + 1$ and the numerical condition ($\Diamond\Diamond$). Let X be the zero locus of a section of E such that X is smooth of dimension n, then for any strictly positive integers $k, l \in \mathbf{N}^*$ with $k + l = n$*

$$\mathrm{Im}\left(\bullet : \mathrm{CH}^k(X)_{\mathbf{Q}} \times \mathrm{CH}^l(X)_{\mathbf{Q}} \to \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q}\cdot h^n,$$

where $h = c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X)$.

2.1.5 A contrasting example

The results of Corollary 2.0.4 ([12]), Corollary 2.0.6 ([75]), and the generalization Theorem 2.1.18 in this chapter suggest the following

Question: To what extent such degeneracy of intersection products in Chow ring can be generalized to other smooth projective varieties?

In this subsection, we construct a contrasting example, showing that in the results above the Calabi-Yau condition is essential, while the complete intersection assumption is not sufficient. More precisely, we will construct a *smooth surface S in $\mathbf{P}^3$ which is of general type, such that the image of the intersection product map*

$$\bullet : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathrm{CH}_0(S)_{\mathbf{Q}}$$

has some elements not proportional to h^2, where $h = c_1(\mathscr{O}_S(1))$.

Let Σ be a general smooth quartic surface in $\mathbf{P}^3 =: \mathbf{P}$. Thus Σ is a K3 surface. Our example S will be the preimage of Σ in $\mathbf{P}' \simeq \mathbf{P}^3$, which is a ramified finite cover of $\mathbf{P}$. The main tool is to consider some nodal curve C in the Severi variety $V_{3,3}$ of Σ, and the ramified cover $\mathbf{P}' \to \mathbf{P}$

is carefully chosen so that the normalization of $\widetilde{C}$ is naturally embedded in $\mathbf{P}'$. Now we give the details of the construction.

Let us denote by $V_{k,g}$ its *Severi variety*:

$$V_{k,g} := \overline{\left\{C \in |\mathscr{O}_\Sigma(k)| : C \text{ is irreducible and nodal with } g(\widetilde{C}) = g\right\}},$$

where $\widetilde{C}$ means the normalization of C, and the closure is taken in $|\mathscr{O}_\Sigma(k)|$. One knows that $V_{k,g}$ is smooth non-empty of dimension g if $0 \leq g \leq 2k^2+1$, whose general member has $\delta := 2k^2 + 1 - g$ nodes (*cf.* [52] [22]).

In particular, $V_{3,2}$ is of dimension 2, and its general member has 17 nodes. We first show that the nodes of the irreducible curves parameterized by $V_{3,2}$ sweep out a 2-dimensional part of Σ. Indeed, consider general members $C_1 \in V_{1,1}$ and $C_2 \in V_{2,1}$, then C_1, C_2 are both irreducible, of normalization genus 1 and they intersect transversally at 8 points. Note that any morphism from an elliptic curve to K3 surface $f : E \to \Sigma$ with nodal image, the normal bundle of f is $\frac{f^*T_\Sigma}{T_E}$, which is clearly trivial. Therefore C_1 and C_2 both vary in a 1-dimensional family with intersection points running over a 2-dimensional part of Σ, and thus the unions $C_1 \cup C_2$ give a 2-dimensional family of (reducible) curves in $|\mathscr{O}_\Sigma(3)|$ with 18 nodes, where at least one node (in $C_1 \cap C_2$) sweeps out a 2-dimensional part of Σ. Keeping this node, we smoothify another node we get a 2-dimensional family of irreducible curves (with 17 nodes) in $V_{3,2}$ with one node sweeping out a 2-dimensional part as desired.

As $\mathrm{CH}_0(\Sigma)_\mathbf{Q}$ is different from $\mathbf{Q}$ and generated by the points of Σ, we can find $C' \in V_{3,2}$ irreducible with 17 nodes $N_1, \cdots N_{17}$, such that at least one of its nodes has class in $\mathrm{CH}_0(\Sigma)_\mathbf{Q}$ different from $c_\Sigma := \frac{1}{4}c_1\left(\mathscr{O}_\Sigma(1)\right)^2$. Quite obviously, there exist thus 16 of them, say the first 16, with sum not rational equivalent to $16c_\Sigma$. Since $V_{3,3}$ is also smooth, containing $V_{3,2}$ as a smooth divisor, we can deform C' in $|\mathscr{O}_\Sigma(3)|$ by keeping the first 16 nodes, and smoothifying the last one, to get $C \in V_{3,3}$ with the sum $Z := N_1 + \cdots + N_{16}$ of its 16 nodes not rationally equivalent to $16c_\Sigma$.

Now we take another copy of $\mathbf{P}^3 =: \mathbf{P}'$, and construct a finite cover $\pi : \mathbf{P}' \to \mathbf{P}$ by taking $[X_0 : X_1 : X_2 : X_3]$ to $[q_0(X) : q_1(X) : q_2(X) : q_3(X)]$ with q_i quadratic polynomials in $X_0, \cdots, X_3$ without base points, the q_i will be given later. We want to have an embedding $\widetilde{C} \hookrightarrow \mathbf{P}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{C} & \dashrightarrow & \mathbf{P}' \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \longrightarrow & \mathbf{P} \end{array} \tag{2.33}$$

We consider the square roots of $n^*\mathscr{O}_C(1)$ in $\mathrm{Jac}(\widetilde{C})$, *i.e.* $L \in \mathrm{Jac}(\widetilde{C})$ such that $L^{\otimes 2} \simeq n^*\mathscr{O}_C(1)$. Note that such L is a degree 6 divisor on the genus 3 curve $\widetilde{C}$. We can choose one of these square roots L which is very ample on $\widetilde{C}$. Indeed, if none of them is very ample, then each square root is of the form $\mathscr{O}_{\widetilde{C}}(K_{\widetilde{C}}+x+y)$ for some $x, y \in \widetilde{C}$. Therefore all the 2-torsion points of $\mathrm{Jac}(\widetilde{C})$ are contained in a translation of the image of $u : \mathrm{Sym}^2\widetilde{C} \to \mathrm{Jac}(\widetilde{C})$, which is again a translation of the theta divisor by Poincaré's formula. However, it is known that for a principally polarized abelian variety, any translation of a theta divisor cannot contain all the 2-torsion points, see for example [51] [53] [42].

L being chosen as above, the corresponding embedding $i : \widetilde{C} \hookrightarrow \mathbf{P}' := \mathbf{P}|L|^* \simeq \mathbf{P}^3$, (with $\mathscr{O}_{\widetilde{C}}(1) = L$), induces a morphism $i^* : H^0(\mathbf{P}', \mathscr{O}_{\mathbf{P}'}(2)) \to H^0(\widetilde{C}, 2L)$. We make the following

Claim(★): for the smoothing $C \in V_{3,3}$ chosen generically, i^* above is an isomorphism.

Since both vector spaces have the same dimension 10, we only need to verify the injectivity of i^*, that is, $\widetilde{C}$ in not contained in a quadric of $\mathbf{P}'$. Suppose on the contrary that there exists a quadric $Q \subset \mathbf{P}'$ containing $\widetilde{C}$, then we have:

Lemma 2.1.31. *$\widetilde{C}$ is hyperelliptic.*

Proof. If $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is smooth, we denote the class of its two fibers by l_1, l_2, and assume $\widetilde{C} = a \cdot l_1 + b \cdot l_2$ for $a, b \in \mathbf{Z}$. Since $\widetilde{C}$ is of degree 6 and $\mathscr{O}_Q(1) = \mathscr{O}_{\mathbf{P}^1}(1) \boxtimes \mathscr{O}_{\mathbf{P}^1}(1)$, we have $a+b = 6$. On the other hand, the fact that $\widetilde{C}$ is of genus 3 implies that $\left(\widetilde{C}^2\right) + \left((-2l_1 - 2l_2) \cdot \widetilde{C}\right) = 4$, which is equivalent to $2ab - 2(a+b) = 4$. Therefore $a = 2, b = 4$ (or $a = 4, b = 2$), *i.e.* $\widetilde{C} \in \left|\mathscr{O}_{\mathbf{P}^1}(2) \boxtimes \mathscr{O}_{\mathbf{P}^1}(4)\right|$. In particular, the projection to the first (or second) ruling of Q shows that $\widetilde{C}$ is hyperelliptic.

If Q is the projective cone over a conic with the singular point O, then $\widetilde{C}$ must pass through O. Indeed, if $O \notin \widetilde{C}$, then $\widetilde{C}$ is a Cartier divisor of Q. Since $\mathrm{Pic}(Q) = \mathbf{Z} \cdot \mathscr{O}_Q(1)$ and $\deg(\widetilde{C}) = 6$, we find that $\widetilde{C}$ is the (smooth) intersection of Q with a cubic. However, the smooth intersection of a cubic and a quadric in $\mathbf{P}^3$ should has genus 4 by the adjunction formula. This contradiction shows $O \in \widetilde{C}$. Then (after the blow-up of Q at O), the projection from O to the conic provides a degree 2 morphism from $\widetilde{C}$ to the conic, showing that $\widetilde{C}$ is hyperelliptic.

If Q is the union of two projective plans, then $\widetilde{C}$ is a plan curve of degree 6, but then its genus should be 10 instead of 3, so this case can not happen at all. □

Since the smoothing C can be chosen in a 3-dimensional family B, and the normalization $\widetilde{C}$ as well as the choice of square root L can also be carried over this base variety B (by shrinking B if necessary), we obtain a 3-dimensional family of hyperelliptic curves mapping to the K3 surface Σ. Their hyperelliptic involutions then yield a family of rational curves on $\Sigma^{[2]}$ parameterized by B, where $\Sigma^{[2]}$ is the Hilbert scheme of 0-dimensional subschemes of length 2 on Σ, which is an irreducible holomorphic symplectic variety (*cf.* [9]). Namely, we have the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & \Sigma^{[2]} \\ \downarrow & & \\ B & & \end{array}$$

where P is a $\mathbf{P}^1$-bundle over B and f is the natural morphism. We now exclude this situation case by case:

- If f is generically finite, or equivalently dominant, then $\Sigma^{[2]}$ would be dominated by a ruled variety, thus uniruled, contradicts to the fact that its canonical bundle is trivial.
- If the image of f is of dimension 3, *i.e.* f dominates a prime divisor D of $\Sigma^{[2]}$. Let $\widetilde{D}$ be a suitable resolution of singularities of D, such that the rationally connected quotient of $\widetilde{D}$ is a morphism $q : \widetilde{D} \to T$. After some blow-ups of P if necessary, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & \widetilde{D} \\ \downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\overline{f}]{} & T \end{array}$$

  Firstly, we note that $\dim(T) \leq 2$ since $\widetilde{D}$ is covered by rational curves. Secondly, since the fibers of q are rationally connected, the morphism $q^* : H^{2,0}(T) \to H^{2,0}(\widetilde{D})$ is surjective. However, let $\sigma \in H^{2,0}(\Sigma^{[2]})$ be the holomorphic symplectic form (*cf.* [9]), then its restriction (more precisely, its pull-back) to $\widetilde{D}$ is non-zero (since $\dim(D) = 3$). Therefore $H^{2,0}(T) \neq 0$, which implies in particular $\dim(T) \geq 2$.

  The above argument shows that $\dim(T) = 2$. As a result, the fibers of q are unions of rational curves, and the dimension of the fibers of $\overline{f}$ is at least 1. Therefore f maps some

1-dimensional family of $\mathbf{P}^1$ into a union of (finitely many) rational curves on $\Sigma^{[2]}$, which implies that all the rational curves in this 1-dimensional family are mapped into a same rational curve, contradicting to the fact that all the rational curves parameterized by B are distinct from each other.

- If the image of f is contained in a surface of $\Sigma^{[2]}$, then a dimension counting shows that through a general point of the surface passes a 1-dimensional family of rational curves, so it is actually a rational surface. As a result, the corresponding points of $\Sigma^{[2]}$ are all equal up to rational equivalence; in other words, the push-forwards of the g^1_2's of the family of hyperelliptic curves, viewed as elements in $\mathrm{CH}_0(\Sigma)_{\mathbf{Q}}$, are all equal. However, for any hyperelliptic curve $\widetilde{C}$ in this family, let $\iota : \widetilde{C} \to \Sigma$ be the composition of the normalization n with the natural inclusion, then

$$\iota_*\left(2g^1_2\right) = \iota_*\left(K_{\widetilde{C}}\right) = \iota_*\left(\iota^*\mathscr{O}_\Sigma(C)|_C - n^*(Z)\right) = \mathscr{O}_\Sigma(3)|_C - 2Z,$$

which is non-constant by our construction, where Z is the sum of the nodes of C. This is a contradiction.

In conclusion, we have proved the claim (★).

Thanks to the isomorphism $i^* : H^0(\mathbf{P}', \mathscr{O}_{\mathbf{P}'}(2)) \to H^0(\widetilde{C}, 2L)$, we can define the morphism $\pi : \mathbf{P}' \to \mathbf{P}$, hence also the $q_0, \cdots, q_3$ as promised, by the following commutative diagram:

$$\begin{array}{ccc}
H^0(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(1)) & \longrightarrow & H^0(C, \mathscr{O}_C(1)) \\
\big\downarrow{\scriptstyle \pi^*} & & \big\downarrow{\scriptstyle n^*} \\
H^0(\mathbf{P}', \mathscr{O}_{\mathbf{P}'}(2)) & \xrightarrow{\ i^*\ } & H^0(\widetilde{C}, 2L)
\end{array}$$

achieving the commutative diagram (2.33). Define $S := \pi^{-1}(\Sigma)$, then $\widetilde{C}$ is a curve in S:

$$\begin{array}{ccccc}
\widetilde{C} & \longrightarrow & S & \longrightarrow & \mathbf{P}' \\
\big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle p} & \square & \big\downarrow{\scriptstyle \pi} \\
C & \longrightarrow & \Sigma & \longrightarrow & \mathbf{P}
\end{array}$$

By the adjunction formula, we have

$$\mathscr{O}_S(\widetilde{C})|_{\widetilde{C}} + K_S|_{\widetilde{C}} = K_{\widetilde{C}} = n^*\Big(\mathscr{O}_\Sigma(C)|_C \otimes \mathscr{O}_C(-Z)\Big).$$

Note that $\deg(S) = \deg(p) = \deg(\pi) = 8$, hence $K_S = \mathscr{O}_S(4) = p^*\mathscr{O}_\Sigma(2)$. Therefore the above equality implies

$$\mathscr{O}_S(\widetilde{C})|_{\widetilde{C}} = n^*\left(\mathscr{O}_C(1) \otimes \mathscr{O}_C(-Z)\right).$$

Pushing this forward to Σ, we deduce that

$$p_*(\widetilde{C}^2) = 12c_\Sigma - 2Z.$$

Since $Z \neq 16c_\Sigma$, $\widetilde{C}^2$ is not proportional to h^2 as desired, where $h = c_1(\mathscr{O}_S(1))$.

In the above construction, we have not yet verified the smoothness of S. However, as above the smoothing C can be chosen in a 3-dimensional family B, and the construction of $\widetilde{C}$, L, π and finally S can also be carried over this base B (by shrinking B if necessary). Generically, the family $\mathscr{S}$ is the transverse pull-back

$$\begin{array}{ccc}
\mathscr{S} & \longrightarrow & \mathbf{P}' \times B \\
\big\downarrow{\scriptstyle p_B} & & \big\downarrow{\scriptstyle \widetilde{\pi}} \\
\Sigma \times B & \longrightarrow & \mathbf{P} \times B
\end{array}$$

which is generically smooth over B, thus a general fiber S is smooth.

2.2 Higher powers of hypersurfaces of general type

We now turn to the study of the smallest diagonal in higher self-products of a non-Fano hypersurface in the projective space. Our goal is to establish Theorem 2.0.8 in the introduction.

First of all, let us fix the basic set-up for this section. Let $\mathbf{P} := \mathbf{P}^{n+1}$ be the projective space, and X be a smooth hypersurface in $\mathbf{P}$ of degree d, with $d \geq n+2$. Thus X is an n-dimensional smooth variety. Since $K_X = \mathscr{O}_X(d-n-2)$, when $d > n+2$, X has ample canonical bundle thus is of general type; when $d = n+2$, X is of Calabi-Yau type and we will recover the results of [75]. Let $k := d+1-n \geq 3$. Our first objective is to express in the Chow group of X^k the class of the *smallest* diagonal

$$\delta_X := \{(x, x, \cdots, x) \mid x \in X\}$$

in terms of bigger diagonals. Since there will be various types of diagonals involved, we need some systematic notation to treat them.

Definition 2.2.1. For any positive integers $s \leq r$,

1. we define the set N_s^r to be the set of all possible *partitions* [8] of r elements into s non-empty *non-ordered* parts. In other words:

$$N_s^r := \{\{1, 2, \cdots, r\} \twoheadrightarrow \{1, 2, \cdots, s\} \text{ surjective maps}\} / \mathfrak{S}_s.$$

   Here the action of the symmetric groups is induced from the action on the target. The action is clearly free, and we take the quotient in the naive (set-theoretical) way. We will view such a surjective map as some sort of *degeneration* of r points into s points. In every equivalence class, we have a canonical representative $\dot{\alpha} : \{1, 2, \cdots, r\} \twoheadrightarrow \{1, 2, \cdots, s\}$ such that $\dot{\alpha}(1), \dot{\alpha}(2), \cdots, \dot{\alpha}(r)$ is alphabetically minimal.

2. For positive integers $t \leq s \leq r$, let $\alpha \in N_s^r$ and $\alpha' \in N_t^s$ be two partitions, then the *composition* $\alpha\alpha' \in N_t^r$ is defined in the natural way, and obviously $\dot{\alpha}' \circ \dot{\alpha}$ is the minimal representative of $\alpha\alpha'$.

3. For every $\alpha \in N_s^r$, and any set (or algebraic variety) Y, we have a natural morphism denoted still by α,

$$\begin{aligned} \alpha : Y^s &\rightarrow Y^r \\ (y_1, y_2, \cdots, y_s) &\mapsto (y_{\dot{\alpha}(1)}, y_{\dot{\alpha}(2)}, \cdots, y_{\dot{\alpha}(r)}) \end{aligned}$$

   where $\dot{\alpha}$ is the minimal representative of α defined above. Note that the morphism induced by their composition $\alpha\alpha'$ is exactly the composition of the morphisms induced by α and α', so there is no ambiguity in the notation $\alpha\alpha' : Y^t \rightarrow Y^r$.

4. The *pull-back* of an r-tuple $\underline{a} = (a_1, a_2, \cdots, a_r)$ by an element $\alpha \in N_s^r$ is defined by

$$\alpha^*(\underline{a}) := (b_1, \cdots, b_s),$$

   where $b_j := \sum_{\dot{\alpha}(i)=j} a_i$ for any $1 \leq j \leq s$. We note that pull-backs are functorial: $(\alpha\alpha')^*(\underline{a}) = \alpha'^*\alpha^*(\underline{a})$.

8. Here we use the terminology ‘partition’ in an unusual way.

Let us explain this definition in a concrete example: if $r = 5, s = 3$, and the partition of $\{1,2,3,4,5\}$ is $\alpha = (\{1,3\};\{4\};\{2,5\}) \in N_3^5$, then the representative $\dot{\alpha} : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 3, 5 \mapsto 2$, and thus the corresponding morphism for any Y is

$$\begin{array}{rcl} \alpha : Y^3 & \to & Y^5 \\ (y_1,y_2,y_3) & \mapsto & (y_1,y_2,y_1,y_3,y_2). \end{array}$$

And also the pull-back of a 5-tuple is $\alpha^*(a_1,\cdots,a_5) = (a_1+a_3, a_2+a_5, a_4)$. If we have another $t = 2$, $\alpha' \in N_2^3$ defined by $(\{1,3\};\{2\})$, then $\alpha\alpha'$ is $(\{1,2,3,5\};\{4\})$, and $\alpha\alpha' : Y^2 \to Y^5$ maps (y_1,y_2) to (y_1,y_1,y_2,y_1,y_1).

This definition is nothing else but all the diagonal inclusions we need in the sequel: for instance, the unique partition in N_1^2 is the diagonal $\Delta_Y \subset Y \times Y$; the three elements in N_2^3 is the so-called *big* diagonals of $Y \times Y \times Y$ in the preceding section, and the unique partition in N_1^r is the smallest diagonal $\delta_Y \subset Y^r$. Note that the morphisms α also induce morphisms from $Y^s\backslash\delta_Y$ to $Y^r\backslash\delta_Y$.

Now return to our geometric setting. As in the preceding section, for any integer $r \geq 2$, we define

$$W_r := \{(y_1,y_2,\cdots,y_r) \in \mathbf{P}^{\times r} \mid y_i \text{ are collinear}\}\backslash\delta_{\mathbf{P}}.$$

In other words, if we denote by $L \to G$ the universal line over the Grassmannian $G := \mathrm{Gr}(\mathbf{P}^1,\mathbf{P})$, then in fact $W_r = \underbrace{L\times_G\cdots\times_G L}_{r}\backslash\delta_L$. In particular W_r is a smooth variety with $\dim W_r = \dim G + r = 2n + r$.

Similarly, for any $r \geq 2$, we define a closed subvariety V_r of $X^r\backslash\delta_X$ by taking the closure in $X^r\backslash\delta_X$ of

$$V_r^o := \{(x_1,x_2,\cdots,x_r) \in X^r \mid x_i \text{ are collinear and distinct}\}$$

$$V_r := \overline{V_r^o}.$$

Then we can prove as in Lemma 2.1.3 (in fact more easily) or as in [75] the following lemma. See also [38].

Lemma 2.2.2. *Consider the intersection of W_r and $X^r\backslash\delta_X$ in $\mathbf{P}^{\times r}\backslash\delta_{\mathbf{P}}$. The intersection scheme has the following irreducible component decomposition:*

$$W_r \cap (X^r\backslash\delta_X) = V_r \cup \bigcup_{2\leq s<r}\bigcup_{\alpha\in N_s^r} \alpha(V_s) \tag{2.34}$$

Moreover, the intersection along each component is transversal, in particular the intersection scheme is of pure dimension $2n$, that is $\dim V_s = 2n$ for any s. In particular, (2.34) also holds scheme-theoretically:

$$\begin{array}{ccc} V_r \cup \bigcup_{2\leq s<r}\bigcup_{\alpha\in N_s^r}\alpha(V_s) & \longrightarrow & X^r\backslash\delta_X \\ \downarrow & \square & \downarrow \\ W_r & \longrightarrow & \mathbf{P}^{\times r}\backslash\delta_{\mathbf{P}} \end{array}$$

We now define some vector bundles on V_r for any $r \in \{2,3,\cdots,k\}$, which are analogues of the vector bundle F defined in the preceding section. Let S be the tautological rank 2 vector bundle on W_r, such that $p : \mathbf{P}(S) \to W_r$ is the tautological $\mathbf{P}^1$-bundle, which admits r tautological sections $\sigma_i : W_r \to \mathbf{P}(S)$, where $i = 1, \cdots, r$. Let $q : \mathbf{P}(S) \to \mathbf{P}^{n+1}$ be the natural morphism. We summarize the situation by the following diagram:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{q} & \mathbf{P} \\ \sigma_i \uparrow \cdots \uparrow \downarrow p & & \\ W_r & & \end{array}$$

Let B_i be the image of the section σ_i, which is a divisor of $\mathbf{P}(S)$, for $i = 1, \cdots, r$. For any r-tuple $\underline{a} := (a_1, a_2, \cdots, a_r)$ such that $\sum_{i=1}^r a_i = k$, we make the following constructions:

1. A sheaf on W_r by

$$\widetilde{F}\left(\underline{a}\right) := p_* \left(q^* \mathscr{O}_{\mathbf{P}}(d) \otimes \mathscr{O}_{\mathbf{P}(S)}(-a_1 B_1 - \cdots - a_r B_r)\right).$$

As in Lemma 2.1.4, we can prove that $\widetilde{F}(\underline{a})$ is a vector bundle on W_r of rank $d + 1 - k = n$, with fiber

$$\widetilde{F}(\underline{a})_{y_1 \cdots y_r} = H^0(\mathbf{P}^1_{y_1 \cdots y_r}, \mathscr{O}(d) \otimes \mathscr{O}(-a_1 y_1 - a_2 y_2 - \cdots - a_r y_r)).$$

2. A rank n vector bundle on V_r by restriction:

$$F(\underline{a}) := \widetilde{F}(\underline{a})|_{V_r}.$$

3. An n-dimensional algebraic cycle on

$$\gamma_{\underline{a}} := \gamma_{a_1, \cdots, a_r} := i_{r*} c_n \left(F(a_1, \cdots, a_r)\right) \in \mathrm{CH}_n(X^r \backslash \delta_X),$$

where for any integer $r \geq 2$, we denote the natural inclusion $i_r : V_r \to X^r \backslash \delta_X$.

Recall that for any $2 \leq s \leq r$ and any $\alpha \in N_s^r$, the morphisms $\alpha : X^s \backslash \delta_X \to X^r \backslash \delta_X$ induces a diagonal inclusion $\alpha : V_s \hookrightarrow W_r$, whose formula is given by repeating some coordinates. We observe the following relation:

Lemma 2.2.3. *For any r-tuple $\underline{a} := (a_1, a_2, \cdots, a_r)$ such that $\sum_{i=1}^r a_i = k$, the restriction of the vector bundle $\widetilde{F}(\underline{a})$ on W_r to the image $\alpha(V_s)$ gives the vector bundle $F(\alpha^*(\underline{a}))$ on V_s, i.e.*

$$\widetilde{F}(\underline{a})|_{\alpha(V_s)} = F(\alpha^*(\underline{a})), \tag{2.35}$$

where $\alpha^*(\underline{a})$ is defined in Definition 2.2.1.

Proof. To avoid heavy notation, let us explain in the simplest case that $s = r - 1$ and the partition α is given by $(\{1, 2\}; \{3\}; \{4\}; \cdots; \{r\})$, then $\alpha : V_s \to W_r$ maps $(x_1, x_2, \cdots, x_s)$ to $(x_1, x_1, x_2, \cdots, x_s)$. Therefore the fiber of $\widetilde{F}(\underline{a})|_{\alpha(V_s)}$ over the point $(x_1, x_2, \cdots, x_s)$ is exactly the fiber of $\widetilde{F}(\underline{a})$ over the point $(x_1, x_1, x_2, \cdots, x_s)$, which is

$$H^0(\mathbf{P}^1_{x_1 \cdots x_s}, \mathscr{O}(d) \otimes \mathscr{O}(-(a_1 + a_2)x_1 - a_3 x_2 - \cdots - a_s x_r)).$$

It is nothing else but the fiber of $F(\alpha^*(\underline{a}))$ over the point $(x_1, x_2, \cdots, x_s) \in V_s$. □

Using the same trick as in Proposition 2.1.7, we obtain the following recursive relations.

Proposition 2.2.4. *The algebraic cycles γ satisfy some recursive equalities: for any integer $r \geq 3$ and any r-tuple $\underline{a} = (a_1, \cdots, a_r)$ with $\sum_{i=1}^r a_i = k$, we have in $\mathrm{CH}_n(X^r \backslash \delta_X)$,*

$$\gamma_{\underline{a}} + \sum_{2\leq s<r} \sum_{\alpha\in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} + P_{\underline{a}}(h_1, \cdots, h_r) = 0, \tag{2.36}$$

where $P_{\underline{a}}$ is a homogeneous polynomial of degree $n(r-1)$ depending on $\underline{a}$. Moreover the starting data are given by

$$\gamma_{a,b} = \prod_{i=0}^{n-1} \Big((b+i)h_1 + (n-1-i+a)h_2 \Big) \tag{2.37}$$

for any $a+b=k$. Here h_i is the pull-back of $h = c_1(\mathscr{O}_X(1))$ by the i-th projection.

Proof. Consider the cartesian square in Lemma 2.2.2:

$$\begin{array}{ccc} V_r \cup \bigcup_{2\leq s<r} \bigcup_{\alpha\in N_s^r} \alpha(V_s) & \xrightarrow{j_3} & X^r \backslash \delta_X \\ \downarrow{\scriptstyle j_4} & \square & \downarrow{\scriptstyle j_2} \\ W_r & \xrightarrow[j_1]{} & \mathbf{P}^{\times r} \backslash \delta_{\mathbf{P}} \end{array}$$

Thanks to Lemma 2.2.2, there is no excess intersection here, we thus obtain:

$$j_2^* j_{1*} c_n \left(\widetilde{F}(\underline{a}) \right) = j_{3*} j_4^* c_n \left(\widetilde{F}(\underline{a}) \right).$$

In the left hand side, $j_{1*} c_n \left(\widetilde{F}(\underline{a}) \right) \in \mathrm{CH}_{n+r}(\mathbf{P}^{\times r} \backslash \delta_{\mathbf{P}}) = \mathrm{CH}_{n+r}(\mathbf{P}^{\times r})$, which can be written as a homogeneous polynomial $-P(H_1, \cdots, H_r)$ of degree $n(r-1)$. Applying j_2^*, H_i restricts to h_i. While in the right hand side,

$$\begin{aligned} j_4^* c_n \left(\widetilde{F}(\underline{a}) \right) &= \sum_{2\leq s\leq r} \sum_{\alpha\in N_s^r} \alpha_* c_n \left(\widetilde{F}(\underline{a})|_{\alpha(V_s)} \right) \\ &= \sum_{2\leq s\leq r} \sum_{\alpha\in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} \quad \text{(by Lemma 2.2.3)} \end{aligned}$$

Putting them together, and noting that N_r^r has only one element inducing the identity map, we have (2.36).

As for the starting data, we only need to calculate $c_n(F(a,b))$. By the same computations in the proof of Lemma 2.1.8, we find that on $V_2 = X \times X \backslash \Delta_X$,

$$F(a,b) = \bigoplus_{i=0}^{n-1} \mathscr{O}_X(i+b) \boxtimes \mathscr{O}_X(n-1-i+a),$$

and the formula (2.37) follows. □

Before we exploit the recursive formula (2.36) further, we need the following easy lemma which allows us to simplify the push-forwards by some diagonal maps. This lemma essentially appeared in [75] (Lemma 3.3), but for the convenience of the readers we give a proof.

Lemma 2.2.5. *Let X be a hypersurface in $\mathbf{P} := \mathbf{P}^{n+1}$ of degree d. Then in $\mathrm{CH}_{n-1}(X \times X)$,*

$$d\Delta_*(h) = (i \times i)^!(\Delta_{\mathbf{P}}) = \sum_{j=1}^{n} h_1^j h_2^{n+1-j},$$

where $i : X \to \mathbf{P}$ is the natural inclusion, $\Delta : X \to X \times X$ is the diagonal inclusion, and h_i is the pull-back of $h := c_1(\mathscr{O}_X(1))$ by the i-th projection.

Proof. Consider the cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ {\scriptstyle i}\downarrow & \square & \downarrow{\scriptstyle i\times i} \\ \mathbf{P} & \xrightarrow[\Delta_{\mathbf{P}}]{} & \mathbf{P}\times\mathbf{P} \end{array}$$

Its excess normal bundle is exactly $N_{X/\mathbf{P}} \simeq \mathscr{O}_X(d)$. Therefore

$$\begin{aligned} d\Delta_*(h) &= \Delta_*\left(c_1\left(\mathscr{O}_X(d)\right)\right) = \Delta_*\left(i^![\mathbf{P}]\cdot c_1(\text{excess normal bundle})\right) \\ &= \Delta_*\left((i\times i)^![\mathbf{P}]\right) = (i\times i)^!(\Delta_{\mathbf{P}}). \end{aligned}$$

Finally, we know that in $\mathrm{CH}_{n+1}(\mathbf{P}\times\mathbf{P})$, we have a decomposition $\Delta_{\mathbf{P}} = \sum_{j=0}^{n+1} H_1^j H_2^{n+1-j}$. Restricting this to $X\times X$, H_i becomes h_i, and $h_i^{n+1} = 0$, proving the lemma. □

For the rest of this section, we will consider only $\mathbf{Q}$-coefficient cycles. Let $c := c_X := \frac{1}{d}h^n \in \mathrm{CH}_0(X)_{\mathbf{Q}}$ be the 0-cycle of degree 1, where $h = c_1\left(\mathscr{O}_X(1)\right)$. Then note that Lemma 2.2.5 implies the following simple equation in $\mathrm{CH}_0(X\times X)_{\mathbf{Q}}$:

$$\Delta_*(c) = c\times c, \tag{2.38}$$

where $\Delta : X \to X\times X$ is the diagonal inclusion.

To make use of the recursive formula (2.36), we need to introduce some terminology and notation. Let $r\geq 2$ be an integer. For any non-empty subset J of $\{1,2,\cdots,r\}$, define the diagonal

$$\Delta_J := \left\{(x_1,\cdots,x_r)\in X^r \mid x_j = x_{j'}, \forall j,j'\in J\right\},$$

which is a cycle of dimension $n(r+1-|J|)$. For example, $\Delta_{\{1,2,\cdots,r\}} = \delta_X$ is the smallest diagonal, and $\Delta_{\{i\}} = X^r$ for any i. Then for any proper subset I of $\{1,2,\cdots,r\}$, we can define the n-dimensional cycle

$$D_I := \Delta_{I^c}\cdot\prod_{i\in I}\mathrm{pr}_i^*\, c, \tag{2.39}$$

where $I^c = \{1,2,\cdots,r\}\backslash I$ is the complementary set. Informally, we could write

$$D_I = \left\{(x_1,\cdots,x_r)\in X^r \mid x_i = c, \forall i\in I; x_j = x_{j'}, \forall j,j'\notin I\right\}. \tag{2.40}$$

For example, $D_\emptyset = \delta_X$; $D_{\{1,2,\cdots,r-1\}} = \underbrace{c\times\cdots\times c}_{r-1}\times X$. And for any i, $D_i := D_{\{i\}}$ is called the i-th *secondary diagonal*. The crucial idea of the calculation of this section is to focus on the coefficients of these secondary diagonals in the expression of γ's.

Definition 2.2.6. An algebraic cycle in $\mathrm{CH}_n(X^r)_{\mathbf{Q}}$ is called of

- *Type A*: if it is a $\mathbf{Q}$-coefficient homogeneous polynomial in $h_1,\cdots,h_r$ of degree $n(r-1)$, such that each h_i appears in every monomial, *i.e.* if it is a $\mathbf{Q}$-linear combination of $\left\{\prod_{j=1}^r h_j^{m_j}; m_j>0, \sum_j m_j = n(r-1)\right\}_{i=1}^r$;

  And for any $0\leq j\leq r-1$:

- *Type B_j*: if it is a $\mathbf{Q}$-linear combination of the cycles D_I for I proper subsets of $\{1,2,\cdots,r\}$ with $|I| = j$.

For example:

- *Type B_0*: if it is a multiple of the cycle $D_\emptyset = \delta_X$. We will not need this type.
- *Type B_1*: if it is a $\mathbf{Q}$-linear combination of the secondary diagonals, *i.e.* the cycles D_i for $1 \leq i \leq r$;
- $\cdots$
- *Type B_{r-1}*: if it is a $\mathbf{Q}$-linear combination of $\left\{\prod_{j\neq i} h_j^n\right\}_{i=1}^r$;

We remark that these notions of types also make sense (except B_0 becomes zero) when viewed as cycles in $\mathrm{CH}_n(X^r\backslash\delta_X)$ by restricting to this open subset. We sometimes write Type $B_{\geq 2}$ for a sum of the form Type B_2 + Type $B_3 + \cdots +$ Type B_{r-1}.

Now we study the behaviors of the various types under push-forwards by diagonal maps. In the simplest case, we have:

Lemma 2.2.7. *Let $r \geq 2$ be an integer and $\alpha \in N_r^{r+1}$ be a partition, inducing a diagonal map $\alpha : X^r\backslash\delta_X \to X^{r+1}\backslash\delta_X$. Then*

1. α_**(Type A) = Type A;*
2. α_**(Type B_j) = Type B_j + Type B_{j+1}, for any $1 \leq j \leq r-1$. In particular,*

$$\alpha_*(\textit{Type } B_1) = \textit{Type } B_1 + \textit{Type } B_2;$$

$$\alpha_*(\textit{Type } B_2) = \textit{Type } B_{\geq 2};$$

Proof. For simplicity, one can suppose $\alpha : (x_1, x_2, \cdots, x_r) \mapsto (x_1, x_1, x_2, \cdots, x_r)$. Then the proofs are just some straightforward computations making use of Lemma 2.2.5 and (2.38). □

Since any $\alpha \in N_s^r$ is a composition of several one-step-degenerations treated in the above lemma, as a first corollary of Proposition 2.2.4, any $(a_1, \cdots, a_r)$ with $\sum_{i=1}^r a_i = k$, $\gamma_{\underline{a}}$ is of the form: Type A + Type $B_1 + \cdots +$ Type B_{r-1}.

We need something more precise: our first objective is to determine the coefficients of secondary diagonals, *i.e.* cycles of type B_1, in the expression of $\gamma_{\underline{a}}$ determined by the recursive relations in Proposition 2.2.4. However, Lemma 2.2.7 tells us that the coefficients of B_1-cycles in the γ's for a certain r are determined only by the coefficients of B_1-cycles in γ's for strictly smaller r's. Now let us work them out.

Proposition 2.2.8. *Let $r \geq 2$ be an integer, $(a_1, \cdots, a_r)$ be an r-tuple with $\sum_{i=1}^r a_i = k$, then*

$$\gamma_{a_1,\cdots,a_r} = \mu_r \sum_{i=1}^r \psi(a_i) D_i + \textit{Type } B_{\geq 2} + \textit{Type } A, \tag{2.41}$$

where the constants $\mu_r = (-1)^r(r-2)!$, and $\psi(a_i) := d \cdot \frac{(n-1+\sum_{j\neq i} a_j)!}{(\sum_{j\neq i} a_j - 1)!} = d \cdot \frac{(d-a_i)!}{(k-1-a_i)!}$.
In particular,

$$\gamma_{1^k} = (-1)^k d! \sum_{i=1}^k D_i + \textit{Type } B_{\geq 2} + \textit{Type } A, \tag{2.42}$$

where $1^k = (\underbrace{1, 1, \cdots, 1}_{k})$.

Proof. Rewrite the recursive formula (2.36): for $r \geq 3$,

$$\gamma_{\underline{a}} + \sum_{2\leq s<r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} = \text{Type } A + \text{Type } B_{r-1} = \text{Type } A + \text{Type } B_{\geq 2};$$

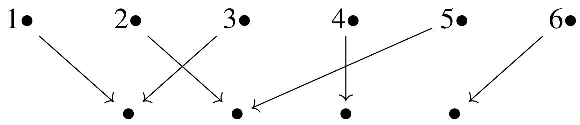
and the starting data (2.37): for any $a + b = k$,

$$\gamma_{a,b} = \frac{(b+n-1)!}{(b-1)!} h_1^n + \frac{(a+n-1)!}{(a-1)!} h_2^n + \text{Type } A = \psi(a) D_1 + \psi(b) D_2 + \text{Type } A.$$

Hence (2.41) holds for $r = 2$. Now we will prove the result by induction on r. Thanks to Lemma 2.2.7, Type A and Type $B_{\geq 2}$ are preserved by α_*, hence we can work throughout this proof 'modulo' these two types, and we will use '$\equiv$' in the place of '$=$' to indicate such simplification.
In the recursive formula

$$\gamma_{\underline{a}} + \sum_{2\leq s<r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} \equiv 0, \tag{2.43}$$

to calculate a typical term $\alpha_* \gamma_{\alpha^*(\underline{a})}$, we first make an elementary remark that we can choose *any* representative of α to define the pull-back and the induced push-forward morphism instead of sticking to the minimal representative as we did before: α_*, α^* compensate each other for the effect of renumbering the parts. An example would be helpful: let $\alpha \in N_4^6$ be as following:



then no matter how one renumbers the four points on the second row, we always have (assuming the induction hypothesis (2.41) for $r = 4$):

$$\begin{aligned} \alpha_* \gamma_{\alpha^*(a_1, \cdots, a_6)} &= \mu_4 \cdot \Big(\psi(a_1 + a_3) D_{\{1,3\}} + \psi(a_2 + a_5) D_{\{2,5\}} + \psi(a_4) D_4 + \psi(a_6) D_6 \Big) \\ &\equiv \mu_4 \cdot \Big(\psi(a_4) D_4 + \psi(a_6) D_6 \Big) \qquad (\mod \text{Type } A + \text{Type } B_{\geq 2}). \end{aligned}$$

The second observation is that the contribution to Type B_1 of a typical term $\alpha_* \gamma_{\alpha^*(\underline{a})}$ with $\alpha \in N_s^r$, is exactly the sum of $\mu_s \cdot \psi(a_i) D_i$ for those i stays 'isolated' in the partition defined by α. One can check this in the above example too, the isolated points are 4 and 6, and the contribution is $\mu_4 \cdot \Big(\psi(a_4) D_4 + \psi(a_6) D_6 \Big)$.
Therefore, the recursive formula (2.43) reads as (by the induction hypothesis for all $2 \leq s < r$):

$$\gamma_{\underline{a}} + \sum_{i=1}^{r} \sum_{2\leq s<r} m_{r,s} \mu_s \cdot \psi(a_i) D_i \equiv 0, \tag{2.44}$$

where $m_{r,s}$ is the cardinality of the set $\{\alpha \in N_s^r \mid i \text{ stays isolated in the partition defined by } \alpha\}$, here $i \in \{1, 2, \cdots, r\}$, and obviously $m_{r,s}$ is independent of i. However, by ignoring the isolated part, it is easy to see that $m_{r,s}$ is exactly the cardinality of N_{s-1}^{r-1}. Therefore to complete the proof, it suffices to show the following identity:

$$\mu_r + \sum_{2\leq s<r} \# N_{s-1}^{r-1} \cdot \mu_s = 0,$$

which is an immediate consequence of the following elementary lemma. □

Lemma 2.2.9. *For any positive integer $m \geq 2$, we have*

$$\sum_{1\leq j\leq m} (-1)^j (j-1)! \cdot \# N_j^m = 0.$$

Proof. For any positive integer $1 \leq j \leq m$, let S_j^m be the set of *surjective* maps from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, j\}$. By definition 2.2.1, N_j^m is the quotient of the action of $\mathfrak{S}_j$ on S_j^m induced by the action on the target. This action is clearly free, thus $\# N_j^m = \frac{1}{j!}\# S_j^m$. Denoting $s_j^m := \# S_j^m$, we have to show the following identity:

$$\sum_{1\leq j\leq m} (-1)^j \frac{1}{j} \cdot s_j^m = 0. \tag{2.45}$$

Now for any integer $1 \leq l \leq m$, we consider the number of all maps from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, l\}$, which is obviously l^m. However on the other hand, we could count this number by classifying these maps by the cardinality of their images: the number of maps with #image= j is exactly $\binom{l}{j} \cdot s_j^m$. Hence,

$$l^m = \sum_{j=1}^{m} \binom{l}{j} \cdot s_j^m.$$

Since this identity holds for $l = 0, 1, \cdots, m$, it is in fact an identity of polynomials of degree m:

$$T^m = \sum_{j=1}^{m} \binom{T}{j} \cdot s_j^m,$$

where T is the variable. Simplifying T from both sides:

$$T^{m-1} = 1 + \sum_{j=2}^{m} s_j^m \cdot \frac{(T-1)(T-2)\cdots(T-j+1)}{j!}.$$

Let $T = 0$, we obtain (2.45), and the lemma follows. □

Now we relate the cycle γ_{1^k} in (2.42) to a geometric constructed cycle. Like before, in the case that X is general, let $F(X)$ be the variety of lines of X. which is a smooth variety of dimension $2n - d - 1 = n - k$ if $k \leq n$, and empty if $k > n$. Still in the case that X is generic, define also the subvariety of X^k

$$\Gamma := \bigcup_{t\in F(X)} \underbrace{\mathbf{P}_t^1 \times \cdots \times \mathbf{P}_t^1}_{k},$$

which is of dimension n if $k \leq n$ and empty if $k > n$.

But we remark that as long as X is smooth, we can always define in a purely intersection theoretical way the *virtual fundamental classes* of $F(X)$ and Γ as algebraic cycles modulo rational equivalence, which in particular is represented by its fundamental class if X is general and $k \leq n$, and zero if $k > n$. More precisely,

$$F(X)^{vir} := c_{d+1}(\mathrm{Sym}^d(S^\vee)) \in \mathrm{CH}_{n-k}(G),$$

which an algebraic cycle supported on the subscheme $F(X)$.

$$\Gamma^{vir} := q_* p^*(F(X)^{vir}) \in \mathrm{CH}_n(X^k),$$

where the morphisms are defined in the following diagram, and S is the universal rank 2 vector bundle on the Grassmannian $G = \mathrm{Gr}(\mathbf{P}^1, \mathbf{P})$.

$$\begin{array}{ccc} \mathbf{P}(S|_{F(X)})^{\times_{F(X)} k} & \xrightarrow[q]{} & X^k \\ \downarrow p & & \\ F(X) & & \end{array}$$

From now on, we will write Γ for Γ^{vir}, the n-dimensional cycle of X^k, satisfying $\Gamma = 0$ when $k > n$.

Write $\Gamma_o := \Gamma|_{X^k \backslash \delta_X} \in \mathrm{CH}_n(X^k \backslash \delta_X)$. As in Lemma 2.1.5,

Lemma 2.2.10. *Let 1^k be the k-tuple $(1, 1, \cdots, 1)$, then $\Gamma_o = \gamma_{1^k}$ in $\mathrm{CH}_n(X^k \backslash \delta_X)$.*

Proof. The defining function of X gives rise to a section of the rank n vector bundle $F(1, 1, \cdots, 1)$ on V_k by restricting to lines. Its zero locus defines exactly the cycle Γ_o in $X^k \backslash \delta_X$. Then the geometrical meaning of top Chern classes proves the desired equality. □

Combining the results of Proposition 2.2.8 and Lemma 2.2.10, we get a decomposition of the class of the smallest diagonal of X^k, except that the multiple λ_0 appearing below could be zero.

Proposition 2.2.11. *There exist rational numbers λ_j for $j = 0, \cdots, k-2$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\mathrm{CH}_n(X^k)_{\mathbf{Q}}$ we have:*

$$\Gamma = \sum_{j=0}^{k-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_k), \tag{2.46}$$

where $\lambda_1 = (-1)^k d!$ is non-zero. More concretely,

$$\Gamma = \lambda_0 \delta_X + (-1)^k d! \sum_{i=1}^{k} D_i + \lambda_2 \sum_{|I|=2} D_I + \cdots + \lambda_{k-2} \sum_{|I|=k-2} D_I + P(h_1, \cdots, h_k). \tag{2.47}$$

Proof. Putting Lemma 2.2.10 into (2.42), we obtain $\Gamma_o = (-1)^k d! \sum_{i=1}^k D_i + \text{Type } B_{\geq 2} + \text{Type } A$ in $\mathrm{CH}_n(X^k \backslash \delta_X)_{\mathbf{Q}}$. Thanks to the localization exact sequence

$$\mathrm{CH}_n(X)_{\mathbf{Q}} \xrightarrow{\delta_*} \mathrm{CH}_n(X^k)_{\mathbf{Q}} \to \mathrm{CH}_n(X^k \backslash \delta_X)_{\mathbf{Q}} \to 0,$$

and the symmetry of Γ, we can write it in the way as stated (remember that Type B_{k-1} is in fact a polynomial of the h_i's). □

A second reflection on (2.46) or (2.47) gives the main result of this section, which is a generalization of Theorem 2.0.5 in the introduction:

Theorem 2.2.12. *Let X be a smooth hypersurface in $\mathbf{P}^{n+1}$ of degree d with $d \geq n+2$. Let $k = d + 1 - n \geq 3$. Then one of the following two cases occurs:*

1. *There exist rational numbers λ_j for $j = 2, \cdots, k-1$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\mathrm{CH}_n(X^k)_{\mathbf{Q}}$ we have:*

$$\delta_X = (-1)^{k-1} \frac{1}{d!} \cdot \Gamma + \sum_{i=1}^{k} D_i + \sum_{j=2}^{k-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_k), \tag{2.48}$$

where D_I is defined in (2.39) or (2.40).
Or

2. *There exist a (smallest) integer* $3 \leq l < k$, *rational numbers* λ_j *for* $j = 2, \cdots, l-2$, *and a symmetric homogeneous polynomial* P *of degree* $n(l-1)$, *such that in* $\mathrm{CH}_n(X^l)_{\mathbf{Q}}$ *we have:*

$$\delta_X = \sum_{i=1}^{l} D_i + \sum_{j=2}^{l-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_l). \tag{2.49}$$

Moreover, $\Gamma = 0$ *if* $d \geq 2n$.

Proof. In (2.46) or (2.47), if $\lambda_0 = -\lambda_1 (= (-1)^{k-1} d!)$ which is non-zero in particular, then we can divide on both sides by λ_1 to get (2.48) in Case 1, up to a rescaling of the numbers λ_i and the polynomial P.

If $\lambda_0 + \lambda_1 \neq 0$, then we project both sides onto the first $k-1$ factors. Since Γ has relative dimension 1 for this projection, it vanishes after the projection. Therefore we get an equality in $\mathrm{CH}_n(X^{k-1})_{\mathbf{Q}}$ of the form:

$$0 = (\lambda_0 + \lambda_1)\delta_X + \lambda'_1 \sum_{i=1}^{k} D_i + \lambda'_2 \sum_{|I|=2} D_I + \cdots + \lambda'_{k-2} \sum_{|I|=k-2} D_I + P'(h_1, \cdots, h_{k-1}).$$

Dividing both sides by $\lambda_0 + \lambda_1$, which is non-zero, we get a decomposition of the smallest diagonal in $\mathrm{CH}_n(X^l)_{\mathbf{Q}}$ for $l = k-1$:

$$\delta_X = \lambda_1 \sum_{i=1}^{l} D_i + \sum_{j=2}^{l-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \cdots, h_l). \tag{2.50}$$

by reverting to the old notation λ_i and P.

If such a decomposition does not exist for $l = k-2$, then by a further projection to the first $k-2$ factors of (2.50), we find $\lambda_1 = 1$ in (2.50) for $l = k-1$. Hence we obtain a decomposition (2.49) in Case 2 for $l = k-1$.

If there does exist such decomposition for $l = k-2$, *i.e.* we have an identity of the form (2.50) for $l = k-2$. By projecting to the first $k-3$ factors, if such decomposition as (2.50) for $l = k-3$ does not exist, then we find $\lambda_1 = 1$ in (2.50) for $l = k-2$. Hence we obtain a decomposition (2.49) in Case 2 for $l = k-2$. If there does exist such decomposition for $l = k-3$, we continue doing the same argument.

Since $H^{n,0}(X) \neq 0$, as in the proof of Lemma 2.1.11, the non-existence of a decomposition of the diagonal $\Delta_X \subset X \times X$ implies that the minimal l for the existence of a decomposition of the form (2.50) is at least 3. Therefore the above argument must stop at some $l \geq 3$, and gives the decomposition (2.49) in Case 2 for this minimal l.

As for the vanishing of Γ, we note that $d \geq 2n$ if and only if $k > n$, in which case we know that Γ is empty. □

Now we draw the following consequence on the ring structure of $\mathrm{CH}^*(X)_{\mathbf{Q}}$ from the above decomposition theorem, generalizing Corollary 2.0.6:

Theorem 2.2.13. *Let* X *be a smooth hypersurface in* $\mathbf{P}^{n+1}$ *of degree* d *with* $d \geq n+2$. *Let* $m = d - n \geq 2$. *Then for any strictly positive integers* $i_1, i_2, \cdots, i_m \in \mathbf{N}^*$ *with* $\sum_{j=1}^{m} i_j = n$, *the image*

$$\mathrm{Im}\left(\mathrm{CH}^{i_1}(X)_{\mathbf{Q}} \times \mathrm{CH}^{i_2}(X)_{\mathbf{Q}} \times \cdots \times \mathrm{CH}^{i_m}(X)_{\mathbf{Q}} \xrightarrow{\bullet} \mathrm{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n$$

Proof. In our notation before, $m = k - 1$. Let $z_j \in \mathrm{CH}^{i_j}(X)_{\mathbf{Q}}$ for $1 \leq j \leq m$. By Theorem 2.2.12, (2.48) or (2.49) holds. Suppose first that we are in Case 1, *i.e.* (2.48). We view its both sides as correspondences from X^m to X. Apply these correspondences to the algebraic cycle $z := z_1 \times \cdots \times z_m \in \mathrm{CH}^n(X^m)_{\mathbf{Q}}$:

- $\delta_{X*}(z) = z_1 \bullet \cdots \bullet z_m$;
- $\Gamma_*(z) = 0$ since $\Gamma_*(z)$ is represented by a linear combination of fundamental classes of certain subvarieties of dimension at least 1, but $\Gamma_*(z)$ is a zero-dimensional cycle, thus vanishes;
- $D_{I*}(z) = 0$ for any $I \neq \{k\}$;
- $D_{k*}(z) = \deg(z_1 \bullet \cdots \bullet z_m) \cdot c_X$;
- $P(h_1, \cdots, h_{m+1})_*(z)$ is always proportional to h^n.

Therefore $z_1 \bullet \cdots \bullet z_m \in \mathbf{Q} \cdot h^{i_1+\cdots+i_m}$.
If we are in Case 2, the same proof goes through. □

Remark 2.2.14. When $d = n + 2$, this recovers the result of [75] as in the first part of the chapter. When $d > n + 2$, the preceding theorem is actually predicted by the Bloch-Beilinson conjecture, which roughly says that $\mathrm{CH}^i(X)_{\mathbf{Q}}$ is controlled by the Hodge structures on the transcendental parts of $H^{2i}(X, \mathbf{Q})$, $H^{2i-1}(X, \mathbf{Q})$, $\cdots$, $H^i(X, \mathbf{Q})$. However by the Lefschetz hyperplane section theorem, the only non-Tate-type Hodge structure of $H^*(X, \mathbf{Q})$ is the middle cohomology $H^n(X, \mathbf{Q})$. Therefore, according to the Bloch-Beilinson conjecture, the smallest i such that $\mathrm{CH}^i(X)_{\mathbf{Q}} \supsetneqq \mathbf{Q} \cdot h^i$ is $\lceil \frac{n}{2} \rceil$. Since in the corollary $\sum_{j=1}^m i_j = n$, the conjecture implies that there is at most one j such that z_j is not proportional to h^{i_j}. Now the above corollary follows from the easy fact that the intersection of any algebraic cycle z with the hyperplane section class h is always $\mathbf{Q}$-proportional to a power of h: write ι for the inclusion of the hypersurface X into the projective space, then $z \bullet dh = \iota^* \iota_*(z)$, which is the pull-back of a cycle of the projective space, thus must be proportional to a power of h.

Chapter 3

Action of symplectic automorphisms on CH_0 of some hyper-Kähler fourfolds

Résumé La variété de Fano des droites d'une hypersurface cubique dans $\mathbf{P}^5$ est une variété symplectique holomorphe irréductible. On démontre que tout automorphisme polarisé symplectique de cette variété agit comme l'identité sur le groupe de Chow des 0-cycles, comme prédit par la conjecture de Bloch-Beilinson.

Abstract We prove that for any polarized symplectic automorphism of the Fano variety of lines of a cubic fourfold (equipped with the Plücker polarization), the induced action on the Chow group of 0-cycles is identity, as predicted by Bloch-Beilinson conjecture.

3.0 Introduction

In this chapter we are interested in an analogue of Bloch's conjecture for the action on 0-cycles of a symplectic automorphism of a irreducible holomorphic symplectic variety. First of all, let us recall Bloch conjecture and the general philosophy of Bloch-Beilinson conjecture which motivate our result.

The Bloch conjecture for 0-cycles on algebraic surfaces states the following (*cf.* [15, Page 17]):

Conjecture 3.0.1 (Bloch)**.** *Let Y be a smooth projective variety, X be a smooth projective surface and $\Gamma \in \mathrm{CH}^2(Y \times X)$ be a correspondence between them. If the cohomological correspondence $[\Gamma]^* : H^{2,0}(X) \to H^{2,0}(Y)$ vanishes, then the Chow-theoretic correspondence*

$$\Gamma_* : \mathrm{CH}_0(Y)_{\mathrm{alb}} \to \mathrm{CH}_0(X)_{\mathrm{alb}}$$

vanishes as well, where $\mathrm{CH}_0(\bullet)_{\mathrm{alb}} := \mathrm{Ker}\,(\mathrm{alb} : \mathrm{CH}_0(\bullet)_{hom} \to \mathrm{Alb}(\bullet))$ *denotes the group of the 0-cycles of degree 0 whose albanese classes are trivial.*

The special case in Bloch's conjecture where $X = Y$ is a surface S and $\Gamma = \Delta_S \in \mathrm{CH}^2(S \times S)$ states: *if a smooth projective surface S admits no non-zero holomorphic 2-forms, i.e. $H^{2,0}(S) = 0$, then* $\mathrm{CH}_0(S)_{\mathrm{alb}} = 0$. This has been proved for surfaces which are not of general type [16], for surfaces rationally dominated by a product of curves (by Kimura's work [46] on the nilpotence conjecture, *cf.* [73, Theorem 2.2.11]), and for Catanese surfaces and Barlow surfaces [64], *etc.* .

What is more related to the present chapter is another particular case of Bloch's conjecture: *let S be a smooth projective surface with irregularity $q = 0$. If an automorphism of finite order f of S*

acts on $H^{2,0}(S)$ as identity, i.e. it preserves any holomorphic 2-forms, then f also acts as identity on $CH_0(S)$. This version is obtained from Conjecture 3.0.1 by taking $X = Y = S$ a surface and $\Gamma = \Delta_S - \Gamma_f \in \mathrm{CH}^2(S \times S)$, where Γ_f is the graph of f. We would like to remark that it is also a consequence of the more general Bloch-Beilinson-Murre conjecture.

Recently Voisin [76] and Huybrechts [39] proved this conjecture for any symplectic automorphism of finite order of a K3 surface (see also [41]):

Theorem 3.0.2 (Voisin, Huybrechts)**.** *Let f be an automorphism of finite order of a projective K3 surface S. If f is symplectic, i.e. $f^*(\omega) = \omega$, where ω is a generator of $H^{2,0}(S)$, then f acts as identity on* $\mathrm{CH}_0(S)$.

The purpose of the chapter is to investigate the analogous results in higher dimensional situation. The natural generalizations of K3 surfaces in higher dimensions are the so-called *irreducible holomorphic symplectic varieties* or *hyperkähler manifolds* (*cf.* [9]), which by definition is a simply connected compact Kähler manifold with $H^{2,0}$ generated by a symplectic form (*i.e.* nowhere degenerate holomorphic 2-form). We can conjecture the following vast generalization of Theorem 3.0.2:

Conjecture 3.0.3. *Let f be an automorphism of finite order of an irreducible holomorphic symplectic projective variety X. If f is symplectic: $f^*(\omega) = \omega$, where ω is a generator $H^{2,0}(X)$. Then f acts as identity on* $\mathrm{CH}_0(X)$.

Like Theorem 3.0.2 is predicted by Bloch's Conjecture 3.0.1, Conjecture 3.0.3 is predicted by the more general Bloch-Beilinson conjecture (*cf.* [13], [6, Chapitre 11], [45], [68, Chapter 11]). Instead of the most ambitious version involving the conjectural category of mixed motives, let us formulate it only for the 0-cycles and in the down-to-earth fashion ([68, Conjecture 11.22]), parallel to Conjecture 3.0.1:

Conjecture 3.0.4 (Bloch-Beilinson)**.** *There exists a decreasing filtration $F^\bullet$ on* $\mathrm{CH}_0(X)_{\mathbf{Q}} := \mathrm{CH}_0(X) \otimes \mathbf{Q}$ *for each smooth projective variety X, satisfying:*

(*i*) $F^0 \mathrm{CH}_0(X)_{\mathbf{Q}} = \mathrm{CH}_0(X)_{\mathbf{Q}}$, $F^1 \mathrm{CH}_0(X)_{\mathbf{Q}} = \mathrm{CH}_0(X)_{\mathbf{Q},hom}$;

(*ii*) *$F^\bullet$ is stable under algebraic correspondences;*

(*iii*) *Given a correspondence* $\Gamma \in \mathrm{CH}^{\dim X}(Y \times X)_{\mathbf{Q}}$. *If the cohomological correspondence* $[\Gamma]^* : H^{i,0}(X) \to H^{i,0}(Y)$ *vanishes, then the Chow-theoretic correspondence* $\mathrm{Gr}^i_F \Gamma_* : \mathrm{Gr}^i_F \mathrm{CH}_0(Y)_{\mathbf{Q}} \to \mathrm{Gr}^i_F \mathrm{CH}_0(X)_{\mathbf{Q}}$ *on the i-th graded piece also vanishes.*

(*iv*) $F^{\dim X+1} \mathrm{CH}_0(X)_{\mathbf{Q}} = 0$.

The implication from the Bloch-Beilinson Conjecture 3.0.4 to Conjecture 3.0.3 is quite straightforward: as before, we take $Y = X$ to be the symplectic variety. If f is of order n, then define two projectors in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbf{Q}}$ by $\pi^{inv} := \frac{1}{n}\left(\Delta_X + \Gamma_f + \cdots + \Gamma_{f^{n-1}}\right)$ and $\pi^{\#} := \Delta_X - \pi^{inv}$. Since $H^{2j-1,0}(X) = 0$ and $H^{2j,0}(X) = \mathbf{C} \cdot \omega^j$, the assumption that f preserves the symplectic form ω implies that $[\pi^{\#}]^* : H^{i,0}(X) \to H^{i,0}(X)$ vanishes for any i. By (*iii*), $\mathrm{Gr}^i_F(\pi^{\#}_*) : \mathrm{Gr}^i_F \mathrm{CH}_0(X)_{\mathbf{Q}} \to \mathrm{Gr}^i_F \mathrm{CH}_0(X)_{\mathbf{Q}}$ vanishes for any i. In other words, for the Chow motive $(X, \pi^{\#})$, $\mathrm{Gr}^i_F \mathrm{CH}_0(X, \pi^{\#}) = 0$ for each i. Therefore by five-lemma and the finiteness condition (*iv*), we have $\mathrm{CH}_0(X, \pi^{\#}) = 0$, that is, $\mathrm{Im}(\pi^{\#}_*) = 0$. Equivalently speaking, any $z \in \mathrm{CH}_0(X)_{\mathbf{Q}}$, $\pi^{inv}(z) = z$, *i.e.* f acts as identity on $\mathrm{CH}_0(X)_{\mathbf{Q}}$. Thanks to Roitman's theorem on the torsion of $\mathrm{CH}_0(X)$, the same still holds true for $\mathbf{Z}$-coefficients.

In [11], Beauville and Donagi provide an example of a 20-dimensional family of 4-dimensional irreducible holomorphic symplectic projective varieties, namely the Fano varieties of lines contained in smooth cubic fourfolds. In this chapter, we propose to study Conjecture 3.0.3 for finite order symplectic automorphisms of this particular family. Our main result is the following:

Theorem 3.0.5. *Let f be an automorphism of a smooth cubic fourfold X. If the induced action on its Fano variety of lines $F(X)$, denoted by $\hat{f}$, preserves the symplectic form, then $\hat{f}$ acts on* $\mathrm{CH}_0(F(X))$ *as identity.*

By Proposition 3.6.1, we can state the above theorem equivalently as: *the polarized symplectic automorphisms of $F(X)$ act as identity on* $\mathrm{CH}_0(F(X))$.

We will show in §3.2 (*cf.* Corollary 3.2.3) how to deduce the above main theorem from the following result:

Theorem 3.0.6 (*cf.* Theorem 3.3.3)**.** *Let f be an automorphism of a smooth cubic fourfold X acting as identity on $H^{3,1}(X)$. Then f acts as the identity on* $\mathrm{CH}_1(X)_{\mathbf{Q}}$.

As a consequence of the main theorem, we will deduce in the last section the following consequence:

Corollary 3.0.7. *Under the same hypothesis as in Theorem 3.0.5: if $\hat{f}$ is a polarized symplectic automorphism of the Fano variety of lines $F(X)$ of a smooth cubic fourfold X, then $\hat{f}$ acts on* $\mathrm{CH}_2(F(X))_{\mathbf{Q},hom}$ *as identity.*

Let us explain the main strategy of of the proof of Theorem 3.0.6: we use the techniques of 'spread' as in Voisin's paper [74]. More precisely, we can summarize as follows the main steps. Let f and X be as in Theorem 3.0.6.

(a) Let $\Gamma_f \subset X \times x$ be the graph of f. Let n be the order of f and $\pi^{inv} := \sum_{i=0}^{n-1} \Gamma_{f^i} \in \mathrm{CH}^4(X \times X)_{\mathbf{Q}}$ be the projector onto the invariant part of X. In order to prove that f acts trivially on $\mathrm{CH}_1(X)_Q$ it suffices to show that there exists a decomposition in $\mathrm{CH}^4(X \times X)_{\mathbf{Q}}$:

$$\Delta_X - \pi^{inv} = \Gamma'_0 + Z' + Z'', \tag{3.1}$$

where Γ'_0 is supported on $Y \times Y$ for a codimension 2 closed algebraic subset $Y \subset X$, and Z', Z'' are the pull-back of cycles on $X \times \mathbf{P}^5$ and $\mathbf{P}^5 \times X$ respectively, *cf.* (3.36).

(b) To prove (3.1), we show firstly that there exists an algebraic cycle Γ'_0, such that $\Delta_X - \pi^{inv} = \Gamma'_0$ has zero cohomology class (with rational coefficients), see Proposition 3.5.2. Now consider the family $\mathscr{X} \to B$ of all smooth cubic fourfolds which are mapped to themselves by the automorphism f (here f is a fixed projective automorphism of $\mathbf{P}^5$). One shows that the cycle Γ'_0 for the varying $X \times X$ fit together to give a cycle Γ' on $\mathscr{X} \times_B \mathscr{X}$, see Proposition 3.5.3. Of course, the cycles $(\Delta_{X_b} - \pi^{inv}_{X_b})$ fit together to give a cycle Γ' on $\mathscr{X} \times_b \mathscr{X}$. Then the cohomology class of $\Gamma - \Gamma'$ restricts to zero on each fiber $X_b \times X_b$. By a Leray spectral sequence argument as in [74], there exist algebraic cycles $\mathscr{Z}'$, $\mathscr{Z}''$ on $\mathscr{X} \times_b \mathscr{X}$, which are the pull-back of cycles on $\mathscr{X} \times \mathbf{P}^5$ and $\mathbf{P}^5 \times \mathscr{X}$ respectively, such that $\Gamma - \Gamma' - \mathscr{Z}' - \mathscr{Z}''$ has zero cohomology class.

(c) Now comes the core of the proof: one show that given $z \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}$ which is homologically trivial there exists a dense open subset $B' \subset B$ such that the restriction of z to the base-changed family $\mathscr{X}' \times_{B'} \mathscr{X}'$ vanishes. There are two main ingredients in the proof: (*i*) One completes the family $\mathscr{X} \times_B \mathscr{X}$ to a smooth projective variety for which the rational equivalence and cohomological equivalence coincide, once we tensor with $\mathbf{Q}$.
(*ii*) To extend a homologically trivial cycle to a homologically trivial cycle in the compactification, we exploit the fact that the Chow motive of a cubic fourfold decomposes into pieces which do not exceed the size of the Chow motives of surfaces.

(d) Applying the result in (*c*) to the cycle $(\Gamma - \Gamma' - \mathscr{Z}' - \mathscr{Z}'')$ gives (3.1) and hence the result.

We organize the chapter as follows. In §3.1, we start describing the parameter space of cubic fourfolds with an action satisfying that the induced actions on the Fano varieties of lines are symplectic. In §3.2, the main theorem is reduced to a statement about the 1-cycles of the cubic fourfold. By varying the cubic fourfold, in §3.3 we reduce the main theorem 3.0.5 to the form that we will prove, which concerns only the 1-cycles of a general member in the family. The purpose of §3.4 is to establish the triviality of Chow groups of some total spaces. The first half §3.4.1 shows the triviality of Chow groups of its compactification; then the second half §3.4.2 passes to the open part by comparing to surfaces. §3.5 proves the main Theorem 3.0.5 by following the same strategy of Voisin's paper [74], which makes use of the result of §3.4. In §3.6 we reformulate the hypothesis in the main theorem to the assumption of being 'polarized'. Finally in §3.7, we verify another prediction of Bloch-Beilinson conjecture on the Chow group of 2-cycles (Corollary 3.0.7) from our main result.

We will work over the complex numbers throughout this chapter.

3.1 Basic settings

In this first section, we establish the basic settings for automorphisms of the Fano variety of a cubic fourfold, and work out the condition corresponding to the symplectic assumption.

Let V be a fixed 6-dimensional $\mathbf{C}$-vector space, and $\mathbf{P}^5 := \mathbf{P}(V)$ be the corresponding projective space of 1-dimensional subspaces of V. Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold, which is defined by a polynomial $T \in H^0(\mathbf{P}^5, \mathscr{O}(3)) = \mathrm{Sym}^3 V^\vee$. Let f be an automorphism of X. Since $\mathrm{Pic}(X) = \mathbf{Z} \cdot \mathscr{O}_X(1)$, any automorphism of X is *induced*: it is the restriction of a linear automorphism of $\mathbf{P}^5$ preserving X, still denoted by f.

Remark 3.1.1. $\mathrm{Aut}(X)$ *is a finite group*, in particular any automorphism of a smooth cubic fourfold is of finite order. In fact, $\mathrm{Aut}(X)$ being a closed subgroup of PGL_6 is an algebraic group of finite type. To show the finiteness, it remains to verify that $\mathrm{Aut}(X)$ is moreover *discrete*, or equivalently, $H^0(X, T_X) = 0$. To show this vanishing property, it requires some (straightforward) calculations. We omit the proof here, and will give a computation-free argument in Remark 3.6.3.

Let f be of order $n \in \mathbf{N}_+$. Since the minimal polynomial of f is semi-simple, we can assume without loss of generality that $f : \mathbf{P}^5 \to \mathbf{P}^5$ is given by:

$$f : [x_0 : x_1 : \cdots : x_5] \mapsto [\zeta^{e_0} x_0 : \zeta^{e_1} x_1 : \cdots : \zeta^{e_5} x_5], \tag{3.2}$$

where $\zeta = e^{\frac{2\pi\sqrt{-1}}{n}}$ is a primitive n-th root of unity and $e_i \in \mathbf{Z}/n\mathbf{Z}$ for $i = 0, \cdots, 5$. Now it is clear that X is preserved by f if and only if its defining equation T is contained in an eigenspace of $\mathrm{Sym}^3 V^\vee$, where $\mathrm{Sym}^3 V^\vee$ is endowed with the induced action coming from V.

Let us make it more precise: as usual, we use the coordinates $x_0, x_1, \cdots, x_5$ of $\mathbf{P}^5$ as a basis of $V^\vee$, then $\left\{\underline{x}^{\underline{\alpha}}\right\}_{\underline{\alpha}\in\Lambda}$ is a basis of $\mathrm{Sym}^3 V^\vee = H^0(\mathbf{P}^5, \mathscr{O}(3))$, where $\underline{x}^{\underline{\alpha}}$ denotes $x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_5^{\alpha_5}$, and

$$\Lambda := \left\{\underline{\alpha} = (\alpha_0, \cdots, \alpha_5) \in \mathbf{N}^5 \mid \alpha_0 + \cdots + \alpha_5 = 3\right\}. \tag{3.3}$$

Therefore the eigenspace decomposition of $\mathrm{Sym}^3 V^\vee$ is the following:

$$\mathrm{Sym}^3 V^\vee = \bigoplus_{j\in\mathbf{Z}/n\mathbf{Z}} \left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C} \cdot \underline{x}^{\underline{\alpha}} \right),$$

where for each $j \in \mathbf{Z}/n\mathbf{Z}$, we define the subset of Λ

$$\Lambda_j := \left\{\underline{\alpha} = (\alpha_0, \cdots, \alpha_5) \in \mathbf{N}^5 \mid {}_{e_0\alpha_0+\cdots+e_5\alpha_5=j \mod n}^{\alpha_0+\cdots+\alpha_5=3}\right\}. \tag{3.4}$$

and the eigenvalue of $\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C} \cdot \underline{x}^{\underline{\alpha}}$ is ζ^j. Therefore, explicitly speaking, we have:

Lemma 3.1.2. *Keeping the notation (3.2), (3.3), (3.4), the cubic fourfold X is preserved by f if and only if there exists a $j \in \mathbf{Z}/n\mathbf{Z}$ such that the defining polynomial $T \in \bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}$.*

Let us deal now with the symplectic condition for the induced action on $F(X)$. First of all, let us recall some basic constructions and facts. The following subvariety of the Grassmannian $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$

$$F(X) := \left\{[L] \in \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5) \mid L \subset X\right\} \tag{3.5}$$

is called the *Fano variety of lines* [1] of X. It is well-known that $F(X)$ is a 4-dimensional smooth projective variety equipped with the restriction of the Plücker polarization of the ambient Grassmannian. Consider the incidence variety (*i.e.* the universal projective line over $F(X)$):

$$P(X) := \{(x, [L]) \in X \times F(X) \mid x \in L\}.$$

We have the following natural correspondence:

$$\begin{array}{ccc} P(X) & \xrightarrow{q} & X\,. \\ \Big\downarrow{\scriptstyle p} & & \\ F(X) & & \end{array}$$

Theorem 3.1.3 (Beauville-Donagi [11])**.** *Using the above notation,*

(*i*) *$F(X)$ is a 4-dimensional irreducible holomorphic symplectic projective variety, i.e. $F(X)$ is simply-connected and $H^{2,0}(F(X)) = \mathbf{C}\cdot\omega$ with ω a nowhere degenerate holomorphic 2-form.*

(*ii*) *The correspondence*

$$p_*q^* : H^4(X, \mathbf{Z}) \to H^2(F(X), \mathbf{Z})$$

is an isomorphism of Hodge structures.

In particular, $p_*q^* : H^{3,1}(X) \xrightarrow{\simeq} H^{2,0}(X)$ is an isomorphism. If X is equipped with an action f as before, we denote by $\hat{f}$ the induced automorphism of $F(X)$. Since the construction of the Fano variety of lines $F(X)$ and the correspondence p_*q^* are both functorial with respect to X, the condition that $\hat{f}$ is *symplectic*, namely $\hat{f}^*(\omega) = \omega$ for ω a generator of $H^{2,0}(F(X))$), is equivalent to the condition that f^* acts as identity on $H^{3,1}(X)$. Working this out explicitly, we arrive at the following

Lemma 3.1.4. *Let f be the linear automorphism in (3.2), and X be a cubic fourfold defined by equation T. Then the followings are equivalent:*

- *f preserves X and the induced action $\hat{f}$ on $F(X)$ is symplectic;*
- *There exists a $j \in \mathbf{Z}/n\mathbf{Z}$ satisfying the equation*

$$e_0 + e_1 + \cdots + e_5 = 2j \mod n, \tag{3.6}$$

such that the defining polynomial $T \in \bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}$, where as in (3.4)

$$\Lambda_j := \left\{\underline{\alpha} = (\alpha_0, \cdots, \alpha_5) \in \mathbf{N}^5 \mid {}^{\alpha_0+\cdots+\alpha_5=3}_{e_0\alpha_0+\cdots+e_5\alpha_5=j \mod n}\right\}.$$

1. In the scheme-theoretic language, $F(X)$ is defined to be the zero locus of $s_T \in H^0\left(\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5), \mathrm{Sym}^3 S^\vee\right)$, where S is the universal tautological subbundle on the Grassmannian, and s_T is the section induced by T using the morphism of vector bundles $\mathrm{Sym}^3 V^\vee \otimes \mathscr{O} \to \mathrm{Sym}^3 S^\vee$ on $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.

Proof. Firstly, the condition that f preserves X is given in Lemma 3.1.2. As is remarked before the lemma, $\hat{f}$ is symplectic if and only if f^* acts as identity on $H^{3,1}(X)$. On the other hand, by Griffiths' theory of the Hodge structures of hypersurfaces (*cf.* [68, Chapter 18]), $H^{3,1}(X)$ is generated by the residue Res $\frac{\Omega}{T^2}$, where $\Omega := \sum_{i=0}^{5}(-1)^i x_i \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_5$ is a generator of $H^0(\mathbf{P}^5, K_{\mathbf{P}^5}(6))$. f given in (3.2), we get $f^*\Omega = \zeta^{e_0+\cdots+e_5}\Omega$ and $f^*(T) = \zeta^j T$. Hence the action of f^* on $H^{3,1}(X)$ is multiplication by $\zeta^{e_0+\cdots+e_5-2j}$, from which we obtain Equation (3.6). □

3.2 Reduction to 1-cycles on cubic fourfolds

The objective of this section is to prove Corollary 3.2.3. It allows us in particular to reduce the main Theorem 3.0.5, which is about the action on 0-cycles on the Fano variety of lines, to the study of the action on 1-cycles of the cubic fourfold itself (see Theorem 3.3.3).

To this end, we want to make use of Voisin's equality (see Proposition 3.2.1(*ii*)) in the Chow group of 0-cycles of the Fano variety of a cubic fourfold. Let X be a (smooth) cubic fourfold, $F := F(X)$ be its Fano variety of lines and $P := P(X)$ be the universal projective line over F fitting into the diagram below:

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array}$$

For any line L contained in X, we denote the corresponding point in F by l. Define $S_l := \{l' \in F \mid L' \cap L \neq \emptyset\}$ to be the surface contained in F parameterizing lines in X meeting a give line L. As algebraic cycles,

$$L = q_* p^*(l) \in \mathrm{CH}_1(X); \tag{3.7}$$

$$S_l = p_* q^*(L) \in \mathrm{CH}_2(F). \tag{3.8}$$

The following relations are discovered by Voisin in [71]:

Proposition 3.2.1. *Let $I := \{(l, l') \in F \times F \mid L \cap L' \neq \emptyset\}$ be the incidence subvariety. We denote by $g \in \mathrm{CH}^1(F)$ the Plücker polarization, and by $c \in \mathrm{CH}^2(F)$ the second Chern class of the restriction to F of the tautological rank 2 subbundle on* $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.

(*i*) *There is a quadratic relation in* $\mathrm{CH}^4(F \times F)$*:*

$$I^2 = \alpha\Delta_F + I \cdot \Gamma + \Gamma',$$

where $\alpha \neq 0$ is an integer, Γ is a degree 2 polynomial in $\mathrm{pr}_1^* g$, $\mathrm{pr}_2^* g$, and Γ' is a weighted degree 4 polynomial in $\mathrm{pr}_1^* g$, $\mathrm{pr}_2^* g$, $\mathrm{pr}_1^* c$, $\mathrm{pr}_2^* c$.

(*ii*) *For any $l \in F$, we have an equality in* $\mathrm{CH}_0(F)$*:*

$$S_l^2 = \alpha \cdot l + \beta S_l \cdot g^2 + \Gamma'', \tag{3.9}$$

where $\alpha \neq 0$ and β are constant integers, Γ'' is a polynomial in g^2 and c of degree 2 with integral coefficients independent of l.

Proof. For the first equality (*i*), *cf.* [71, Proposition 3.3]. For (*ii*), we restrict the relation in (*i*) to a fiber $\{l\} \times F$, then $I|_{\{l\}\times F} = S_l$ and $\Delta_F|\{l\} \times F = l$. Hence the equation (3.9). □

Corollary 3.2.2. *Given an automorphism f of a cubic fourfold X, let L be a line contained in X and $l \in \mathrm{CH}_0(F)$, $S_l \in \mathrm{CH}_2(F)$ be the cycles as above. Then the followings are equivalent:*

(*i*) $\hat{f}(l) = l$ *in* $\mathrm{CH}_0(F)$ *;*

(*ii*) $f(L) = L$ *in* $\mathrm{CH}_1(X)$ *;*

(*iii*) $\hat{f}(S_l) = S_l$ *in* $\mathrm{CH}_2(F)$.

The same equivalences hold also for Chow groups with rational coefficients.

Proof. (*i*) ⇒ (*ii*): by (3.7) and the functorialities of p and q.
(*ii*) ⇒ (*iii*): by (3.8) and the functorialities of q and p.
(*iii*) ⇒ (*i*): by (3.9) and the fact that g, c are all invariant by $\hat{f}$, we obtain $\alpha\left(l - \hat{f}(l)\right) = 0$ in $\mathrm{CH}_0(F)$ with $\alpha \neq 0$. However by Roitman theorem $\mathrm{CH}_0(F)$ is torsion-free, thus $l = \hat{f}(l)$ in $\mathrm{CH}_0(F)$.
Of course, the same [2] proof gives the same equivalences for Chow groups with rational coefficients.
□

In particular, we have:

Corollary 3.2.3. *Let f be an automorphism of a cubic fourfold X and F be the Fano variety of lines of X, equipped with the induced action $\hat{f}$. Then the followings are equivalent:*

(*i*) $\hat{f}$ *acts on* $\mathrm{CH}_0(F)$ *as identity;*

(*ii*) $\hat{f}$ *acts on* $\mathrm{CH}_0(F)_{\mathbf{Q}}$ *as identity;*

(*iii*) f *acts on* $\mathrm{CH}_1(X)$ *as identity;*

(*iv*) f *acts on* $\mathrm{CH}_1(X)_{\mathbf{Q}}$ *as identity.*

Proof. Since Paranjape [59] proves [3] that $\mathrm{CH}_1(X)$ is generated by the lines contained in X, the previous corollary gives the equivalences (*i*) ⇔ (*iii*) and (*ii*) ⇔ (*iv*). On the other hand, F is simply-connected and thus its Albanese variety is trivial. Therefore $\mathrm{CH}_0(F)$ is torsion-free by Roitman's theorem, hence (*i*) ⇔ (*ii*). □

We remark that this corollary allows us to reduced Theorem 3.0.5 to Theorem 3.0.6 which is stated purely in terms of the action on the Chow group of the 1-cycles of the cubic fourfold itself.

3.3 Reduction to the general member of the family

Our basic approach to the main theorem 3.0.5 is to *vary* the cubic fourfold in family and make use of certain good properties of the total space (*cf.* §3.4) to get some useful information for a member of the family. To this end, we give in this section a family version of previous constructions, and then by combining Corollary 3.2.3, we reduce the main theorem 3.0.5 to Theorem 3.3.3 which is a statement for 1-cycles of a *general* member in the family.

Fix $n \in \mathbf{N}_+$, fix f as in (3.2) and fix a solution $j \in \mathbf{Z}/n\mathbf{Z}$ of (3.6). Consider the projective space parameterizing certain possibly singular cubic hypersurfaces in $\mathbf{P}^5$.

$$\overline{B} = \mathbf{P}\left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}\right),$$

2. In fact easier, because we do not need to invoke Roitman theorem.
3. For rational coefficients it can be easily deduced by the argument in [32].

where Λ_j is defined in (3.4). Let us denote the universal family by

$$\begin{array}{c} \overline{\mathscr{X}} \\ \big\downarrow{\scriptstyle \pi} \\ \overline{B} \end{array}$$

whose fibre over the a point $b \in \overline{B}$ is a cubic hypersurface in $\mathbf{P}^5$ denoted by X_b. Let $B \subset \overline{B}$ be the Zariski open subset parameterizing the smooth ones. By base change, we have over B the universal family of smooth cubic fourfolds with a (constant) fiberwise action f, and similarly the universal Fano variety of lines $\mathscr{F}$ equipped with the corresponding fiberwise action $\hat{f}$:

$$\begin{array}{ccccc} \hat{f} \circlearrowright \mathscr{F} & & \mathscr{X} \hookrightarrow B \times \mathbf{P}^5 \circlearrowleft f \\ & \searrow & \big\downarrow{\scriptstyle \pi} \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & & B & \end{array} \tag{3.10}$$

The fibre over $b \in B$ of $\mathscr{F}$ is denoted by $F_b = F(X_b)$, on which $\hat{f}$ acts symplectically by construction.

By the following general fact, we claim that to prove the main theorem 3.0.5, it suffices to prove it for a very general member in the family:

Lemma 3.3.1. *Let $\mathscr{F} \to B$ be a smooth projective fibration with a fibrewise action $\hat{f}$ (for example in the situation (3.10) before). If for a general point $b \in B$, $\hat{f}$ acts as identity on $\mathrm{CH}_0(F_b)$, then the same thing holds true for any $b \in B$.*

Proof. For any $b_0 \in B$, we want to show that $\hat{f}$ acts as identity on $\mathrm{CH}_0(F_{b_0})$. Given any 0-cycle $Z \in \mathrm{CH}_0(F_{b_0})$, we can find a base-change (by taking B' to be successive general hyperplane sections of $\mathscr{F}$):

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \big\downarrow & \square & \big\downarrow \\ B' & \longrightarrow & B \end{array}$$

and a cycle $\mathscr{Z} \in \mathrm{CH}_{\dim B'}(\mathscr{F}')$, such that $\mathscr{Z}|_{F'_{b'_0}} = Z$, where $b'_0 \in B'$ is a preimage of $b_0 \in B$, hence $F'_{b'_0} = F_{b_0}$. Now consider $\Gamma := \hat{f}^*\mathscr{Z} - \mathscr{Z} \in \mathrm{CH}_{\dim B'}(\mathscr{F}')$, by assumption it satisfies $\Gamma|_{F'_{b'}} = 0$ in $\mathrm{CH}_0(F'_{b'})$ for a general point $b' \in B'$. However, by an argument of Hilbert scheme (*cf.* [68, Chapter 22]), the set of points $b' \in B'$ satisfying $\Gamma|_{F'_b} = 0$ is a countable union of closed algebraic subsets. Thus together with another countably many proper closed algebraic subsets, they cover B'. By Baire theorem, in this countable collection there exists one which is in fact the entire B', *i.e.* $\Gamma|_{F'_{b'}} = 0$ holds for every $b' \in B'$. In particular, for b'_0, we have $\hat{f}^*Z - Z = \Gamma|_{F_{b_0}} = \Gamma|_{F'_{b'_0}} = 0$. □

Remark 3.3.2. Thanks to this lemma, instead of defining B as the parameter space of smooth cubic fourfolds, we can and we will feel free to shrink B to any of its Zariski open subsets whenever we want to in the rest of the chapter.

To summarize this section, we reduce the main theorem 3.0.5 into the following statement:

Theorem 3.3.3. *Let $n = p^m$ be a power of a prime number, f be an automorphism of $\mathbf{P}^5$ given by (3.2):*

$$f : [x_0 : x_1 : \cdots : x_5] \mapsto [\zeta^{e_0}x_0 : \zeta^{e_1}x_1 : \cdots : \zeta^{e_5}x_5],$$

and $j \in \mathbf{Z}/n\mathbf{Z}$ *be a solution to (3.6):* $e_0 + e_1 + \cdots + e_5 = 2j \mod n$.
If for a general point $b \in \overline{B} := \mathbf{P}\left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}\right)$, X_b *is smooth, then* f *acts as identity on* $\mathrm{CH}_1(X_b)_{\mathbf{Q}}$, *where*

$$\Lambda_j := \left\{\underline{\alpha} = (\alpha_0, \cdots, \alpha_5) \in \mathbf{N}^5 \mid {}^{\alpha_0+\cdots+\alpha_5=3}_{e_0\alpha_0+\cdots+e_5\alpha_5=j \mod n}\right\}$$

as in (3.4).

Theorem 3.3.3 $\Rightarrow$ *Theorem 3.0.5.* First of all, in order to prove the main theorem 3.0.5, we can assume that the order of f is a power of a prime number: suppose the prime factorization of the order of f is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

Let $g_i = f^{\frac{n}{p_i^{a_i}}}$ for any $1 \le i \le r$, then g_i is of order $p_i^{a_i}$. Since $\hat{f}$ acts symplecticly on $F(X)$, so do the g_i's. Then by assumption, $\hat{g}_i$ acts as identity on $\mathrm{CH}_0(F(X))$ for any i. Finally, by Chinese remainder theorem, there exist $b_1, \cdots, b_r \in \mathbf{N}$ such that $f = \prod_{i=1}^r g_i^{b_i}$. Therefore, $\hat{f} = \prod_{i=1}^r \hat{g}_i^{b_i}$ acts as identity on $\mathrm{CH}_0(F(X))$ as well. Secondly, the parameter space $\overline{B}$ comes from the constraints we deduced in Lemma 3.1.4. Thirdly, Lemma 3.3.1 allows us to reduce the statement to the case of a (very) general member in the family. Finally, we can switch from $\mathrm{CH}_0(F_b)$ to $\mathrm{CH}_1(X_b)_{\mathbf{Q}}$ by Corollary 3.2.3. □

3.4 The Chow group of the total space

As a step toward the proof of Theorem 3.3.3, we establish in this section the following result.

Proposition 3.4.1. *Consider the direct system consisting of the open subsets* B *of* $\overline{B}$*, then we have*

$$\varinjlim_{B} \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q},hom} = 0.$$

More precisely, for an open subset B *of* $\overline{B}$*, and for any homologically trivial codimension 4* $\mathbf{Q}$*-cycle* z *of* $\mathscr{X} \times_B \mathscr{X}$*, there exists a dense open subset* $B' \subset B$*, such that the restriction of* z *to the base changed family* $\mathscr{X}' \times_{B'} \mathscr{X}'$ *is rationally equivalent to 0.*

We achieve this in two steps: the first one is to show that homological equivalence and rational equivalence coincide on a resolution of singularities of the compactification $\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}}$ (see Proposition 3.4.2 below); in the second step, to pass to the open variety $\mathscr{X} \times_B \mathscr{X}$, we need to 'extend' a homologically trivial cycle of the open variety to a cycle homologically trivial of the compactification (see Proposition 3.4.3 below). More precisely, let B be an open subset of $\overline{B}$:

Proposition 3.4.2 (Step 1)**.** *There exists a resolution of singularities* $\tau : W \to \overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}}$*, such that the rational equivalence and homological equivalence coincide on* W *when tensored with* $\mathbf{Q}$ *(see Definition 3.4.4 below). In particular,* $\mathrm{CH}^4(W)_{\mathbf{Q},hom} = 0$.

Proposition 3.4.3 (Step 2)**.** *For any homologically trivial cycle* $z \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q},hom}$*, there exist a dense open subset* $B' \subset B$ *and a* homologically trivial *cycle* $\overline{z} \in \mathrm{CH}^4(\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}})_{\mathbf{Q},hom}$*, such that*

$$z|_{\mathscr{X}'\times_{B'}\mathscr{X}'} = \overline{z}|_{\mathscr{X}'\times_{B'}\mathscr{X}'} \in \mathrm{CH}^4(\mathscr{X}' \times_{B'} \mathscr{X}')_{\mathbf{Q}}, \tag{3.11}$$

where $\mathscr{X}'$ *is the base changed family over* B'.

Proposition 3.4.2 and Proposition 3.4.3 will be proved in Subsection §3.4.1 and Subsection §3.4.2 respectively. Admitting them, we first give the proof of Proposition 3.4.1:

Prop. 3.4.2+Prop. 3.4.3 ⇒ Prop. 3.4.1. Let $B', \mathscr{X}', z, \bar{z}, \tau, W$ be as in the previous propositions. Let

$$W^o := W \times_{\overline{B}} B'$$

be the open subset of W over B', and $\tau^o : W^o \to \mathscr{X}' \times_{B'} \mathscr{X}'$ be the base change of τ. Since $\bar{z}$ is homologically trivial, so is $\tau^*(\bar{z}) \in \mathrm{CH}^4(W)_{\mathbf{Q},hom}$. However, by Proposition 3.4.2, $\tau^*(\bar{z})$ is (rationally equivalent to) zero, hence so is its restriction to the open subset W^o. Therefore by (3.11) and the projection formula,

$$z|_{\mathscr{X}'\times_{B'}\mathscr{X}'} = \bar{z}|_{\mathscr{X}'\times_{B'}\mathscr{X}'} = \tau^o_*(\tau^*\bar{z}|_{W^o}) = 0.$$

□

3.4.1 The Chow group of the compactification

In this subsection, we prove Proposition 3.4.2.

We first recall the following notion due to Voisin [74, §2.1]:

Definition 3.4.4. We say a smooth projective variety X satisfies *property $\mathscr{P}$*, if the cycle class map is an isomorphism

$$[-] : \mathrm{CH}^*(X)_{\mathbf{Q}} \xrightarrow{\cong} H^*(X, \mathbf{Q}).$$

Here we provide some examples and summarize some operations that preserve this property $\mathscr{P}$. For details, *cf.* [74].

Lemma 3.4.5. (*i*) *Homogeneous variety of the form G/P satisfies property $\mathscr{P}$, where G is a linear algebraic group and P is a parabolic subgroup. For example, projective spaces, Grassmannians, flag varieties, etc. .*

(*ii*) *If X and Y satisfy property $\mathscr{P}$, then so does $X \times Y$.*

(*iii*) *If X satisfies property $\mathscr{P}$, and E is a vector bundle on it, then the projective bundle $\mathbf{P}(E)$ satisfies property $\mathscr{P}$.*

(*iv*) *If X satisfies property $\mathscr{P}$, and $Z \subset X$ is a smooth subvariety satisfying property $\mathscr{P}$, then so is the blow up variety $\mathrm{Bl}_Z X$.*

(*v*) *Let $f : X \to X'$ be a surjective generic finite morphism. If X satisfies property $\mathscr{P}$, then so does X'.*

Since some toric geometry will be needed in the sequel, let us also recall some standard definitions and properties, see [33], [25] for details. Given a lattice N and a fan Δ in $N_{\mathbf{R}}$, one can construct a toric variety of dimension rank(N), which will be denoted by $X(\Delta)$. By definition, $X(\Delta)$ is the union of affine toric varieties $\mathrm{Spec}(\mathbf{C}[N^\vee \cap \sigma^\vee])$, where $N^\vee$ is the dual lattice, $\sigma^\vee$ is the dual cone in $N^\vee_{\mathbf{R}}$ and σ runs over the cones in Δ. A fan Δ is said *regular* if each cone in Δ is generated by a part of a $\mathbf{Z}$-basis of N. Let N' be another lattice and Δ' be a fan in $N'_{\mathbf{R}}$. Then a homomorphism (as abelian groups) $f : N \to N'$ induces a rational map of the toric varieties $\phi : X(\Delta) \dashrightarrow X(\Delta')$. Such maps are called *equivariant* or *monomial*.

Proposition 3.4.6. *Using the above notation for toric geometry, then we have:*

(*i*) *$X(\Delta)$ is smooth if and only if Δ is regular.*

(*ii*) *$\phi : X(\Delta) \dashrightarrow X(\Delta')$ is a morphism if and only if for any cone $\sigma \in \Delta$, there exists a cone $\sigma' \in \Delta'$ such that f sends σ into σ'.*

(*iii*) *Any fan admits a refinement consisting of regular cones.*

(*iv*) *Any smooth projective toric variety satisfies property $\mathscr{P}$.*

(*v*) *ϕ admits an elimination of indeterminacies:*

$$\begin{array}{ccc} X(\widetilde{\Delta}) & & \\ \downarrow \tau & \searrow^{\widetilde{\phi}} & \\ X(\Delta) & \overset{\phi}{\dashrightarrow} & X(\Delta') \end{array}$$

such that $X(\widetilde{\Delta})$ is smooth projective satisfying property $\mathscr{P}$.

Proof. For (*i*), [25, Theorem 3.1.19]; for (*ii*), [25, Theorem 3.3.4]; for (*iii*), [25, Theorem 11.1.9]; for (*iv*), [25, Theorem 12.5.3]. Finally, (*v*) is a consequence of the first four: by (*iii*), we can find a regular refinement of $\Delta \cup f^{-1}(\Delta')$, denoted by $\widetilde{\Delta}$, then $X(\widetilde{\Delta})$ is smooth by (*i*) and satisfies property $\mathscr{P}$ by (*iv*). Moreover, (*ii*) implies that $\phi \circ \tau : X(\widetilde{\Delta}) \to X(\Delta')$ is a morphism. □

Turning back to our question, we adopt the previous notation as in Theorem 3.3.3.

We can view $\overline{B} = \mathbf{P}\left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C} \cdot \underline{x}^{\underline{\alpha}}\right)$ as an *incomplete* linear system on $\mathbf{P}^5$ associated to the line bundle $\mathscr{O}_{\mathbf{P}^5}(3)$. We remark that by construction in §3.1, each member of $\overline{B}$ (which is a possibly singular cubic fourfold) is preserved under the action of f. Consider the rational map associated to this linear system:

$$\phi := \phi_{|\overline{B}|} : \mathbf{P}^5 \dashrightarrow \overline{B}^{\vee},$$

where $\overline{B}^{\vee}$ is the dual projective space consisting of the hyperplanes of $\overline{B}$. We remark that since $\overline{B}^{\vee}$ has a basis given by monomials, the above rational map ϕ is a *monomial* map between two toric varieties (*cf.* the definition before Proposition 3.4.6).

Lemma 3.4.7. (*i*) *There exists an elimination of indeterminacies of ϕ:*

$$\begin{array}{ccc} \widetilde{\mathbf{P}^5} & & \\ \tau \downarrow & \searrow^{\widetilde{\phi}} & \\ \mathbf{P}^5 & \overset{\phi}{\dashrightarrow} & \overline{B}^{\vee} \end{array}$$

such that $\widetilde{\mathbf{P}^5}$ is smooth projective satisfying property $\mathscr{P}$.

(*ii*) *The strict transform of $\overline{\mathscr{X}} \subset \mathbf{P}^5 \times \overline{B}$, denoted by $\widetilde{\overline{\mathscr{X}}}$, is the incidence subvariety in $\widetilde{\mathbf{P}^5} \times \overline{B}$:*

$$\widetilde{\overline{\mathscr{X}}} = \left\{(x, b) \in \widetilde{\mathbf{P}^5} \times \overline{B} \mid b \in \widetilde{\phi}(x)\right\}.$$

Proof. (*i*) By Proposition 3.4.6(*v*).
(*ii*) follows from the fact that for $x \in \mathbf{P}^5$ not in the base locus of $\overline{B}$, $b \in \phi(x)$ if and only if $(x, b) \in \overline{\mathscr{X}}$. □

Corollary 3.4.8. *$\widetilde{\overline{\mathscr{X}}}$ is smooth satisfies property $\mathscr{P}$.*

Proof. Thanks to Lemma 3.4.7(*iii*), $\widetilde{\overline{\mathscr{X}}} \subset \widetilde{\mathbf{P}^5} \times \overline{B}$ is the incidence subvariety with respect to $\widetilde{\phi} : \widetilde{\mathbf{P}^5} \to \overline{B}^{\vee}$. Therefore the first projection $\widetilde{\overline{\mathscr{X}}} \to \widetilde{\mathbf{P}^5}$ is a projective bundle (whose fiber over $x \in \widetilde{\mathbf{P}^5}$ is the hyperplane of $\overline{B}$ determined by $\widetilde{\phi}(x) \in \overline{B}^{\vee}$), hence smooth. By Lemma 3.4.5(*iii*), $\widetilde{\overline{\mathscr{X}}}$ satisfies property $\mathscr{P}$. □

We remark that the action of f on $\mathbf{P}^5$ lifts to $\widetilde{\mathbf{P}^5}$ because the base locus of $\overline{B}$ is clearly preserved by f. Correspondingly, the linear system $\overline{B}$ pulls back to $\widetilde{\mathbf{P}^5}$ to a base-point-free linear system, still denoted by $\overline{B}$, with each member preserved by f, and the morphism $\widetilde{\phi}$ constructed above is exactly the morphism associated to this linear system.

To deal with the (possibly singular) variety $\overline{\mathscr{X}}\times_{\overline{B}}\overline{\mathscr{X}}$, we follow the same recipe as before (see Lemma 3.4.9, Lemma 3.4.10 and Proposition 3.4.2). The morphism $\widetilde{\phi}\times\widetilde{\phi}:\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}\to\overline{B}^\vee\times\overline{B}^\vee$ induces a rational map

$$\varphi:\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}\dashrightarrow \mathrm{Bl}_{\Delta_{\overline{B}^\vee}}\left(\overline{B}^\vee\times\overline{B}^\vee\right).$$

We remark that this rational map φ is again monomial, simply because $\phi:\mathbf{P}^5\dashrightarrow\overline{B}^\vee$ is so.

Lemma 3.4.9. *There exists an elimination of indeterminacies of φ:*

$$\begin{array}{ccc} \widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}} & \xrightarrow{\ \widetilde{\varphi}\ } & \mathrm{Bl}_\Delta(\overline{B}^\vee\times\overline{B}^\vee) \\ {\scriptstyle\tau}\downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5} & \xrightarrow{\ \widetilde{\phi}\times\widetilde{\phi}\ } & \overline{B}^\vee\times\overline{B}^\vee \end{array}$$

such that $\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}$ is smooth projective satisfying property $\mathscr{P}$.

Proof. It is a direct application of Proposition 3.4.6(v). □

Consider the rational map $\overline{B}^\vee\times\overline{B}^\vee\dashrightarrow\mathrm{Gr}(\overline{B},2)$ defined by 'intersecting two hyperplanes', where $\mathrm{Gr}(\overline{B},2)$ is the Grassmannian of codimension 2 sub-projective spaces of $\overline{B}$. Blowing up the diagonal will resolve the indeterminacies:

$$\begin{array}{ccc} \mathrm{Bl}_{\Delta_{\overline{B}^\vee}}\left(\overline{B}^\vee\times\overline{B}^\vee\right) & & \\ \downarrow & \searrow & \\ \overline{B}^\vee\times\overline{B}^\vee & \dashrightarrow & \mathrm{Gr}(\overline{B},2) \end{array}$$

Composing it with $\widetilde{\varphi}$ constructed in the previous lemma, we obtain a morphism

$$\psi:\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}\to\mathrm{Gr}(\overline{B},2).$$

As in Lemma 3.4.7, we have

Lemma 3.4.10. *Consider the following incidence subvariety of $\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}\times\overline{B}$ with respect to ψ:*

$$W:=\left\{(z,b)\in\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}\times\overline{B}\mid b\in\psi(z)\right\}.$$

(i) *The first projection $W\to\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}$ is a projective bundle, whose fiber over $z\in\widetilde{\widetilde{\mathbf{P}^5}\times\widetilde{\mathbf{P}^5}}$ is the codimension 2 sub-projective space determined by $\psi(z)\in\mathrm{Gr}(\overline{B},2)$.*

(ii) *W has a birational morphism onto $\overline{\mathscr{X}}\times_{\overline{B}}\overline{\mathscr{X}}$.*

Proof. (i) is obvious.
(ii) Firstly we have a morphism $W\to\mathbf{P}^5\times\mathbf{P}^5\times\overline{B}$. We claim that the image is exactly $\overline{\mathscr{X}}\times_{\overline{B}}\overline{\mathscr{X}}$: since for two general points x_1,x_2 in $\mathbf{P}^5$, $\psi(x_1,x_2)=\phi(x_1)\cap\phi(x_2)$, thus $(x_1,x_2,b)\in W$ is by definition equivalent to $b\in\phi(x_1)\cap\phi(x_2)$, which is equivalent to $(x_1,x_2,b)\in\overline{\mathscr{X}}\times_{\overline{B}}\overline{\mathscr{X}}$. □

Now we can accomplish our first step of this section:

Proof of Proposition 3.4.2. Since W is a projective bundle (Lemma 3.4.10(i)) over the variety $\widetilde{\mathbf{P}^5 \times \mathbf{P}^5}$ satisfying property $\mathscr{P}$ (Lemma 3.4.9(ii)), W satisfies also property $\mathscr{P}$ (Lemma 3.4.5(iii)). Then we conclude by Lemma 3.4.10(ii). □

3.4.2 Extension of homologically trivial algebraic cycles

In this subsection we prove Proposition 3.4.3.

To pass from the compactification $\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}}$ to the space $\mathscr{X} \times_B \mathscr{X}$ which concerns us, we would like to mention Voisin's 'conjecture N' ([74, Conjecture 0.6]):

Conjecture 3.4.11 (Conjecture N)**.** *Let X be a smooth projective variety, and let $U \subset X$ be an open subset. Assume an algebraic cycle $Z \in \mathrm{CH}^i(X)_{\mathbf{Q}}$ has cohomology class $[Z] \in H^{2i}(X, \mathbf{Q})$ which vanishes when restricted to $H^{2i}(U, \mathbf{Q})$. Then there exists another cycle $Z' \in \mathrm{CH}^i(X)_{\mathbf{Q}}$, which is supported on $X \backslash U$ and such that $[Z'] = [Z] \in H^{2i}(X, \mathbf{Q})$.*

This Conjecture N implies the following conjecture (*cf.* [73, Lemma 3.3.5]):

Conjecture 3.4.12. *Let X be a smooth projective variety, and U be an open subset of X. If $\mathrm{CH}^i(X)_{\mathbf{Q},hom} = 0$, then $\mathrm{CH}^i(U)_{\mathbf{Q},hom} = 0$.*

According to this conjecture, Proposition 3.4.2 would have implied the desired result Proposition 3.4.1. To get around this conjecture N, our starting point is the following observation in [74, Lemma 1.1]. For the sake of completeness, we recall the proof in [74].

Lemma 3.4.13. *Conjecture N is true for $i \leq 2$. In particular, for $i \leq 2$ and for any $Z^o \in \mathrm{CH}^i(U)_{\mathbf{Q},hom}$, there exists $W \in \mathrm{CH}^i(X)_{\mathbf{Q},hom}$ such that $W|_U = Z^o$.*

Proof. Since $[Z]|_U = 0$ in $H^{2i}(U, \mathbf{Q})$, we have $[Z] \in \mathrm{Im}\left(H^{2i}_Y(X, \mathbf{Q}) \to H^{2i}(X, \mathbf{Q})\right)$. Let $\widetilde{Y} \to Y$ be a resolution of singularities of Y, then by mixed Hodge theory $[Z] \in \mathrm{Im}\left(H^{2i-2c}(\widetilde{Y}, \mathbf{Q}) \to H^{2i}(X, \mathbf{Q})\right)$, where $c \geq 1$ is the codimension of Y. Because $[Z]$ is a Hodge class and $H^{2i-2c}(\widetilde{Y}, \mathbf{Q})$ is a polarized pure Hodge structure, there exists a Hodge class $\gamma \in H^{2i-2c}(\widetilde{Y}, \mathbf{Q})$ which maps to $[Z]$. As $2i - 2c \leq 2$, γ must be algebraic, and we thus obtain an algebraic cycle on $\widetilde{Y}$. Pushing it forward to Y, we get the cycle Z' as wanted. As for the claim on Z^o, we simply take Z to be the closure of Z^o, construct Z' as before, and finally take W to be $Z - Z'$. □

Now the crucial observation is that *the Chow motive of a cubic fourfold does not exceed the size of Chow motives of surfaces*, so that we can reduce the problem to a known case of Conjecture N, namely Lemma 3.4.13. To illustrate, we first investigate the situation of one cubic fourfold (absolute case), then make the construction into the family version.

Absolute case:

Let X be a (smooth) cubic fourfold. Recall the following diagram as in the proof of the unirationality of cubic fourfold:

$$\begin{array}{ccc} \mathbf{P}(T_X|_L) & \overset{q}{\dashrightarrow} & X \\ \downarrow & & \\ L & & \end{array}$$

Here we fix a line L contained in X, and the vertical arrow is the natural $\mathbf{P}^3$-fibration, and the rational map q is defined in the following classical way: for any $(x, v) \in \mathbf{P}(T_X|_L)$ where v is a non-zero tangent vector of X at $x \in L$, then as long as the line $\mathbf{P}(\mathbf{C} \cdot v)$ generated by the tangent vector v is not contained in X, the intersection of this line $\mathbf{P}(\mathbf{C} \cdot v)$ with X should be three (not necessarily distinct) points with two of them x. Let y be the remaining intersection point. We define $q : (x, v) \mapsto y$. By construction, the indeterminacy locus of q is $\{(x, v) \in \mathbf{P}(T_X|_L) \mid \mathbf{P}(\mathbf{C} \cdot v) \subset X\}$. Note that q is dominant of degree 2.

By Hironaka's theorem, we have an elimination of indeterminacies:

$$\begin{array}{ccc} \mathbf{P}\,\widetilde{(T_X|_L)} & & \\ \downarrow{\scriptstyle \tau} & \searrow^{\widetilde{q}} & \\ \mathbf{P}\,(T_X|_L) & \overset{q}{\dashrightarrow} & X \\ \downarrow & & \\ L & & \end{array}$$

where τ is the composition of a series of successive blow ups along smooth centers of dimension ≤ 2, and $\widetilde{q}$ is surjective thus generically finite (of degree 2).

We follow the notation of [6] to denote the category of Chow motives with rational coefficients by $\mathrm{CHM}_{\mathbf{Q}}$, and to write $\mathfrak{h}$ for the Chow motive of a smooth projective variety, which is a *contravariant* functor

$$\mathfrak{h} : \mathrm{SmProj}^{op} \to \mathrm{CHM}_{\mathbf{Q}}\,.$$

Denote $M := \mathbf{P}\,(T_X|_L)$ and $\widetilde{M} := \mathbf{P}\,\widetilde{(T_X|_L)}$. Let S_i be the blow up centers of τ and $c_i = \operatorname{codim} S_i \in \{2, 3, 4\}$. By the blow up formula and the projective bundle formula for Chow motives (*cf.* [6, 4.3.2]),

$$\mathfrak{h}\left(\widetilde{M}\right) = \mathfrak{h}\left(\mathbf{P}\,(T_X|_L)\right) \oplus \bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l) = \left(\bigoplus_{l=0}^{3} \mathfrak{h}(L)(-l)\right) \oplus \left(\bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l)\right),$$

and since $L \simeq \mathbf{P}^1$,

$$\mathfrak{h}\left(\widetilde{M}\right) = \left(\mathbb{1} \oplus \mathbb{1}(-1)^{\oplus 2} \oplus \mathbb{1}(-2)^{\oplus 2} \oplus \mathbb{1}(-3)^{\oplus 2} \oplus \mathbb{1}(-4)\right) \oplus \left(\bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l)\right), \tag{3.12}$$

where $\mathbb{1} := \mathfrak{h}(\mathrm{pt})$ is the trivial motive. On the other hand, since $\widetilde{q}_*\widetilde{q}^* = \deg(\widetilde{q}) = 2 \cdot \mathrm{id}$, $\mathfrak{h}(X)$ is a direct factor of $\mathfrak{h}\left(\mathbf{P}\,\widetilde{(T_X|_L)}\right)$, which has been decomposed in (3.12). This gives a precise explanation of what we mean by saying that $\mathfrak{h}(X)$ *does not exceed the size of motives of surfaces* above.

By the monoidal structure of $\mathrm{CHM}_{\mathbf{Q}}$ (*cf.* [6, 4.1.4]), the motive of $\widetilde{M} \times \widetilde{M}$ has the following form:

$$\mathfrak{h}\left(\widetilde{M} \times \widetilde{M}\right) = \bigoplus_{k \in J} \mathfrak{h}(V_k \times V'_k)(-l_k), \tag{3.13}$$

where J is the index set parameterizing all possible products, and $V_k \times V'_k$ is of one of the following forms:

- $\mathrm{pt} \times \mathrm{pt}$ and $l_k = 0$ or 1;
- $\mathrm{pt} \times S_i$ or $S_i \times \mathrm{pt}$ and $l_k = 1$;
- $l_k \geq 2$.

For each summand $\mathfrak{h}(V_k \times V'_k)(-l_k)$ in (3.13), the inclusion of this direct factor

$$\iota_k \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbf{Q}}}\left(\mathfrak{h}(V_k \times V'_k)(-l_k), \mathfrak{h}\left(\widetilde{M} \times \widetilde{M}\right)\right) \tag{3.14}$$

determines a natural correspondence from $V_k \times V'_k$ to $\widetilde{M} \times \widetilde{M}$. Similarly, for each $k \in J$, the projection to the k-th direct factor

$$p_k \in \mathrm{Hom}_{\mathrm{CHM}_{\mathbf{Q}}}\left(\mathfrak{h}\left(\widetilde{M} \times \widetilde{M}\right), \mathfrak{h}(V_k \times V'_k)(-l_k)\right) \tag{3.15}$$

determines also a natural correspondence from $\widetilde{M} \times \widetilde{M}$ back to $V_k \times V'_k$. By construction

$$p_k \circ \iota_k = \mathrm{id} \in \mathrm{End}_{\mathrm{CHM}_{\mathbf{Q}}}\left(\mathfrak{h}(V_k \times V'_k)(-l_k)\right), \quad \text{for any } \ k \in J;$$

$$\sum_{k \in J} \iota_k \circ p_k = \mathrm{id} \in \mathrm{End}_{\mathrm{CHM}_{\mathbf{Q}}}\left(\mathfrak{h}(\widetilde{M} \times \widetilde{M})\right).$$

Equivalently, the last equation says

$$\sum_{k \in J} \iota_k \circ p_k = \Delta_{\widetilde{M} \times \widetilde{M}} \ \text{ in } \mathrm{CH}^*(\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M})_{\mathbf{Q}}. \tag{3.16}$$

Construction in family:

We now turn to the family version of the above constructions. To this end, we need to choose a specific line for each cubic fourfold in the family, and also a specific point on the chosen line. Therefore a base change (*i.e.* $T \to B$ constructed below) will be necessary to construct the family version of the previous p_k and ι_k's (see Lemma 3.4.14).

Precisely, consider the universal family $\mathscr{X}$ of cubic fourfolds over B, and the universal family of Fano varieties of lines $\mathscr{F}$ as well as the universal incidence varieties $\mathscr{P}$:

$$\begin{array}{ccccc} & & \mathscr{P} & & \\ & \swarrow & & \searrow & \\ \mathscr{F} & & & & \mathscr{X} \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

By taking general hyperplane sections of $\mathscr{P}$, we get T a subvariety of it, such that the induced morphism $T \to B$ is generically finite. In fact, by shrinking[4] B (and also T correspondingly), we can assume $T \to B$ is finite and étale, and hence T is smooth.

By base change construction, we have over T a universal family of cubic fourfolds $\mathscr{Y}$, a universal family of lines $\mathscr{L}$ contained in $\mathscr{Y}$ and a section $\sigma : T \to \mathscr{L}$ corresponding to the universal family of the chosen points in $\mathscr{L}$. We summarize the situation in the following diagram:

$$\begin{array}{ccccc} \mathscr{L} & \hookrightarrow & \mathscr{Y} & \xrightarrow{\ r\ } & \mathscr{X} \\ {\scriptstyle \sigma}\nwarrow\!\!\searrow & & \downarrow{\scriptstyle \pi'} & \square & \downarrow{\scriptstyle \pi} \\ & & T & \longrightarrow & B \end{array} \tag{3.17}$$

where for any $t \in T$ with image b in B, the fiber $Y_t = X_b$, L_t is a line contained in it and $\sigma(t) \in L_t$. As $T \to B$ is finite and étale, so is $r : \mathscr{Y} \to \mathscr{X}$.

4. Recall that we are allowed to shrink B whenever we want, see Remark 3.3.2.

Now we define

$$\mathscr{M} := \mathbf{P}\left(T_{\mathscr{Y}/T}|_{\mathscr{L}}\right),$$

and a dominant rational map of degree 2

$$q : \mathscr{M} \dashrightarrow \mathscr{Y} .$$

Over $t \in T$, the fiber of $\mathscr{M}$ is $M_t = \mathbf{P}\left(T_{Y_t}|_{L_t}\right)$, and the restriction of q to this fiber is exactly the rational map $\mathbf{P}\left(T_{Y_t}|_{L_t}\right) \dashrightarrow Y_t$ constructed before in the absolute case.

By Hironaka's theorem, we have an elimination of indeterminacies of q:

$$\begin{array}{ccc} \widetilde{\mathscr{M}} & & \\ \downarrow{\scriptstyle\tau} & \searrow^{\widetilde{q}} & \\ \mathscr{M} & \overset{q}{\dashrightarrow} & \mathscr{Y} \\ \downarrow & & \swarrow \\ \mathscr{L} & & \\ \downarrow & \swarrow & \\ T & & \end{array} \tag{3.18}$$

such that, up to shrinking B (and also T correspondingly), τ consists of blow ups along smooth centers which are *smooth* over T (by generic smoothness theorem) of relative dimension (over T) at most 2. Suppose the blow up centers are $\mathscr{S}_i$, whose codimension is denoted by $c_i \in \{2,3,4\}$. A fiber of $\widetilde{\mathscr{M}}$, $\mathscr{M}$, $\mathscr{S}_i$ is exactly $\widetilde{M}$, M, S_i respectively constructed in the absolute case. In the same fashion, we denote by $\mathscr{V}_k$ and $\mathscr{V}'_k$ the family version of the varieties V_k and V'_k in the absolute case, which is nothing else but of the form $\mathscr{S}_i \times_T \mathscr{S}_j$ or $T \times_T \mathscr{S}_i$ *etc.* . Let d_k (*resp.* d'_k) be the dimension of V_k (*resp.* V'_k), *i.e.* when $V_k = \mathrm{pt}$, $\mathscr{V}_k = \sigma(T)$ and $d_k = 0$; when $V_k = S_i$, $\mathscr{V}_k = \mathscr{S}_i$ and $d_k = 4 - c_i$, similarly for $\mathscr{V}'_k$.

We can now globalize the correspondences (3.14) and (3.15) into their following family versions. Here we use the same notation:

Lemma 3.4.14. (i) *For any $k \in J$, there exist natural correspondences (over T)*

$$\iota_k \in \mathrm{CH}^{d_k+d'_k+l_k}\left(\mathscr{V}_k \times_T \mathscr{V}'_k \times_T \widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}}\right)_{\mathbf{Q}},$$

$$p_k \in \mathrm{CH}^{8-l_k}\left(\widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}} \times_T \mathscr{V}_k \times_T \mathscr{V}'_k\right)_{\mathbf{Q}},$$

such that the following two identities hold on each fiber: for any $t \in T$, we have

$$(p_k \circ \iota_k)_t = \Delta_{V_{k,t} \times V'_{k,t}} \quad \textit{for any } k \in J;$$

$$\sum_{k \in J} (\iota_k \circ p_k)_t = \Delta_{\widetilde{M}_t \times \widetilde{M}_t}. \tag{3.19}$$

(ii) *Up to shrinking B and T correspondingly, we have in* $\mathrm{CH}^*(\widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}})_{\mathbf{Q}}$,

$$\sum_{k \in J} \iota_k \circ p_k = \Delta_{\widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}}}. \tag{3.20}$$

Proof. (*i*). For the existence, it suffices to remark that the correspondences ι_k and p_k can in fact be universally defined over T, because when we make the canonical isomorphisms (3.12) or (3.13) precise by using the projective bundle formula and blow up formula, they are just compositions of inclusions, pull-backs, intersections with the relative $\mathscr{O}(1)$ of projective bundles, each of which can be defined in family over T. Note that in this step of the construction, we used the section σ to made the isomorphism $\mathfrak{h}(L_t) \simeq \mathbb{1} \oplus \mathbb{1}(-1)$ well-defined in family over T, because this isomorphism amounts to choose a point on the line.
Finally, the equality (3.19) is exactly the equality (3.16) in the absolute case.
(*ii*). The equation (3.20) is a direct consequence of (3.19), thanks to the Bloch-Srinivas type argument of spreading rational equivalences (*cf.* [18], [68, Corollary 10.20]). □

Keeping the notation in Diagram (3.17) and Diagram (3.18), we consider the generic finite morphism

$$\widetilde{q} \times \widetilde{q} : \widetilde{\mathscr{M}} \times_T \widetilde{\mathscr{M}} \to \mathscr{Y} \times_T \mathscr{Y},$$

and the finite étale morphism

$$r \times r : \mathscr{Y} \times_T \mathscr{Y} \to \mathscr{X} \times_B \mathscr{X}.$$

For each $k \in J$, composing the graphs of these two morphisms with ι_k, we get a correspondence over B from $\mathscr{V}_k \times_T \mathscr{V}'_k$ to $\mathscr{X} \times_B \mathscr{X}$:

$$\Gamma_k := \Gamma_{r\times r} \circ \Gamma_{\widetilde{q}\times\widetilde{q}} \circ \iota_k \in \mathrm{CH}^{d_k+d'_k+l_k}\left(\mathscr{V}_k \times_T \mathscr{V}'_k \times_B \mathscr{X} \times_B \mathscr{X}\right)_{\mathbf{Q}}; \tag{3.21}$$

similarly, composing p_k with the transposes of their graphs, we obtain a correspondence over B in the other direction:

$$\Gamma'_k := p_k \circ \left({}^t\Gamma_{\widetilde{q}\times\widetilde{q}}\right) \circ \left({}^t\Gamma_{r\times r}\right) \in \mathrm{CH}^{8-l_k}\left(\mathscr{X} \times_B \mathscr{X} \times_B \mathscr{V}_k \times_T \mathscr{V}'_k\right)_{\mathbf{Q}}. \tag{3.22}$$

Lemma 3.4.15. *The sum of compositions of the above two correspondences satisfies:*

$$\sum_{k\in J} \Gamma_k \circ \Gamma'_k = 4\deg(T/B)\cdot \mathrm{id},$$

as correspondences from $\mathscr{X} \times_B \mathscr{X}$ to itself.

Proof. It is an immediate consequence of the equation (3.20) and the projection formula (note that $\deg(r \times r) = \deg(T/B)$ and $\deg(\widetilde{q} \times \widetilde{q}) = 4$). □

For any $k \in J$, fix a smooth projective compactification $\overline{\mathscr{V}_k \times_T \mathscr{V}'_k}$ of $\mathscr{V}_k \times_T \mathscr{V}'_k$ such that the composition $\overline{\mathscr{V}_k \times_T \mathscr{V}'_k} \dashrightarrow T \to B \to \overline{B}$ is a morphism. Recall that $\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}}$ is a (in general singular) compactification of $\mathscr{X} \times_B \mathscr{X}$. Taking the closure of the two correspondences Γ_k and Γ'_k, we obtain correspondences between the compactifications:

$$\overline{\Gamma_k} \in \mathrm{CH}^{d_k+d'_k+l_k}\left(\overline{\mathscr{V}_k \times_T \mathscr{V}'_k} \times_{\overline{B}} \overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}}\right)_{\mathbf{Q}};$$

$$\overline{\Gamma'_k} \in \mathrm{CH}^{8-l_k}\left(\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{V}_k \times_T \mathscr{V}'_k}\right)_{\mathbf{Q}}.$$

For some technical reasons in the proof below, we have to separate the index set J into two parts and deal with them differently. Recall that below the equation (3.13), we observed that there are three types of elements $(V_k \times V'_k, l_k)$ in J. Define the subset consisting of elements of the third type:

$$J' := \left\{k \in J \mid (V_k \times V'_k, l_k) \text{ satisfies } l_k \geq 2\right\}.$$

And define J'' to be the complement of J': elements of the first two types. Note that for any $k \in J''$, the corresponding $(V_k \times V'_k, l_k)$ satisfies always

$$\dim(V_k \times V'_k) < 4 - l_k \quad \text{for any } \ k \in J''. \tag{3.23}$$

We can now accomplish the main goal of this subsection:

Proof of Proposition 3.4.3. Let $z \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q},hom}$. To simplify the notation, we will omit the lower star for the correspondences Γ_k, Γ'_k, $\overline{\Gamma}_k$ and $\overline{\Gamma}'_k$ since we never use their transposes.

An obvious remark: when $k \in J''$, for any $b \in B$, the fiber $\left(\Gamma'_k\right)_t(z_b) \in \mathrm{CH}^{4-l_k}(V_{k,t} \times V'_{k,t})_{\mathbf{Q}} = 0$ by dimension reason (*cf.* (3.23)), thus

$$\left(\sum_{k\in J''} \Gamma_k \circ \Gamma'_k(z)\right)_b = 0. \tag{3.24}$$

As a result, for any $b \in B$, in $\mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$,

$$\left(\sum_{k\in J'} \Gamma_k\left(\Gamma'_k(z)\right)\right)_b = \left(\sum_{k\in J} \Gamma_k\left(\Gamma'_k(z)\right)\right)_b = 4\deg(T/B)\cdot z_b, \tag{3.25}$$

where the first equality comes from (3.24) and the second equality is by Lemma 3.4.15.

On the other hand, $\Gamma'_k(z) \in \mathrm{CH}^{4-l_k}\left(\mathscr{V}_k \times_T \mathscr{V}'_k\right)_{\mathbf{Q},hom}$ is homologically trivial. We claim that for any $k \in J'$, the cycle $\Gamma'_k(z)$ 'extends' to a *homologically trivial* algebraic cycle in the compactification, *i.e.* there exists $\xi_k \in \mathrm{CH}^{4-l_k}\left(\overline{\mathscr{V}_k \times_T \mathscr{V}'_k}\right)_{\mathbf{Q},hom}$ such that $\xi_k|_{\mathscr{V}_k\times_B\mathscr{V}'_k} = \Gamma'_k(z)$. Indeed, since $4 - l_k \leq 2$ for $k \in J'$ and $\mathscr{V}_k \times_T \mathscr{V}'_k$ is smooth by construction, we can apply Lemma 3.4.13 to find ξ_k.

Now for any $k \in J'$, let us consider the cycle $\overline{\Gamma_k}(\xi_k) \in \mathrm{CH}^4(\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}})_{\mathbf{Q},hom}$. Its fiber over a point $b \in B$ is:

$$\left(\overline{\Gamma_k}(\xi_k)\right)_b = \left(\Gamma_k(\xi_k|_{\mathscr{V}_k\times_B\mathscr{V}'_k})\right)_b = \left(\Gamma_k \circ \Gamma'_k(z)\right)_b.$$

Therefore by (3.25), the restrictions of the two cycles

$$\beta := \sum_{k\in J'} \overline{\Gamma_k}(\xi_k) \ \text{ and } \ 4\deg(T/B)\cdot z$$

are the same in $\mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$ for any $b \in B$. Again by the argument of Bloch and Srinivas(*cf.* [18], [68, §10.2], [73, §2]), there exists a dense open subset $B' \subset B$, such that

$$(4\deg(T/B)z)\,|_{\mathscr{X}'\times_{B'}\mathscr{X}'} = \beta|_{\mathscr{X}'\times_{B'}\mathscr{X}'} \in \mathrm{CH}^4(\mathscr{X}' \times_{B'} \mathscr{X}')_{\mathbf{Q}},$$

where $\mathscr{X}'$ is the base changed family over B'. Defining

$$\bar{z} = \frac{1}{4\deg(T/B)}\beta \in \mathrm{CH}^4(\overline{\mathscr{X}} \times_{\overline{B}} \overline{\mathscr{X}})_{\mathbf{Q}},$$

to conclude Proposition 3.4.3, it suffices to remark that $\bar{z}$ is by construction homologically trivial: since the ξ_k's are homologically trivial, so is $\beta = \sum_{k\in J'} \overline{\Gamma_k}(\xi_k)$ and hence $\bar{z}$. □

3.5 Proof of Theorem 3.3.3

The content of this section is the proof of Theorem 3.3.3.

We keep the notation n, f, j, Λ_j, $\overline{B}$ as in the statement of Theorem 3.3.3. Let B be an open subset of $\overline{B}$ parameterizing smooth cubic fourfolds. In the following diagram,

$$\begin{array}{ccc} f_b \circlearrowright X_b & \hookrightarrow & \mathscr{X} \circlearrowleft f \\ \downarrow & \square & \downarrow \pi \\ b & \hookrightarrow & B \end{array} \tag{3.26}$$

π is the universal family equipped with a fiberwise action f; for each $b \in B$, we write f_b the restriction of f on the cubic fourfold X_b if we need to distinguish it from f. By construction, for any $b \in B$, f_b is an automorphism of X_b of order n which acts as identity on $H^{3,1}(X_b)$.

To begin with, we study the Hodge structures of the fibers:

Lemma 3.5.1. *For any $b \in B$,*

(i) *f^* is an order n automorphism of the Hodge structure $H^4(X_b, \mathbf{Q})$ and $f_* = (f^*)^{-1} = (f^*)^{n-1}$.*

(ii) *There is a direct sum decomposition into sub-Hodge structures*

$$H^4(X_b, \mathbf{Q}) = H^4(X_b, \mathbf{Q})^{inv} \oplus^{\perp} H^4(X_b, \mathbf{Q})^{\#}, \tag{3.27}$$

where the first summand is the f^*-invariant part, and the second summand is its orthogonal complement with respect to the intersection product $< -, - >$.

(iii) *$H^{3,1}(X_b) \subset H^4(X_b, \mathbf{C})^{inv}$.*

(iv) *$H^4(X_b, \mathbf{Q})^{\#}$ is generated by the classes of some codimension 2 algebraic cycles.*

Proof. (i) is obvious since f^* must preserve the Hodge structure. The last equality comes from $f_* f^* = f^* f_* = \mathrm{id}$ and $(f^*)^n = \mathrm{id}$.
(ii). Since f^* is of finite order (see Remark 3.1.1), it is semi-simple: $H^4(X_b, \mathbf{Q})$ decomposes as direct sum of eigenspaces, where $H^4(X_b, \mathbf{Q})^{inv}$ corresponds to eigenvalue 1 and $H^4(X_b, \mathbf{Q})^{\#}$ is the sum of other eigenspaces. Moreover, f^* preserves the intersection pairing $< -, - >$, thus the invariant eigenspace is orthogonal to any other eigenspace: for any cohomology classes x satisfying $f^*(x) = x$ and y satisfying $f^*(y) = \lambda y$ with $\lambda \neq 1$, then $< x, y >=< f^*x, f^*y >= \lambda < x, y >$, thus $< x, y >= 0$.
(iii). This is our assumption that f^* acts as identity on $H^{3,1}(X)$.
(iv). By (iii), $H^4(X_b, \mathbf{Q})^{\#} \subset H^{3,1}(X_b)^{\perp} = H^{2,2}(X_b)$, *i.e.* $H^4(X_b, \mathbf{Q})^{\#}$ is generated by rational Hodge classes of degree 4. However, the Hodge conjecture is known to be true for cubic fourfolds ([78]). We deduce that $H^4(X_b, \mathbf{Q})^{\#}$ is generated by the classes of some codimension 2 subvarieties in X_b. □

Define the algebraic cycle

$$\pi^{inv} := \frac{1}{n}\left(\Delta_{\mathscr{X}} + \Gamma_f + \cdots + \Gamma_{f^{n-1}}\right) \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}. \tag{3.28}$$

Here π^{inv} can be viewed as a family of self-correspondences of X_b parameterized by B, more precisely:

$$\pi^{inv}|_{X_b \times X_b} =: \pi_b^{inv} = \frac{1}{n}\left(\Delta_{X_b} + \Gamma_{f_b} + \cdots + \Gamma_{f_b^{n-1}}\right) \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}. \tag{3.29}$$

It is now clear that π_b^{inv} acts on $H^4(X_b, \mathbf{Q})$ by:

$$[\pi_b^{inv}]^* = \frac{1}{n}\left(Id + f_b^* + \cdots + \left(f_b^*\right)^{n-1}\right),$$

which is exactly the orthogonal projector onto the invariant part in the direct sum decomposition (3.27). On the other hand, π_b^{inv} acts as identity on $H^0(X_b, \mathbf{Q})$, $H^2(X_b, \mathbf{Q})$, $H^6(X_b, \mathbf{Q})$, $H^8(X_b, \mathbf{Q})$.

Define another cycle

$$\Gamma := \Delta_{\mathscr{X}} - \pi^{inv} \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}. \tag{3.30}$$

Then $\Gamma_b := \Gamma|_{X_b \times X_b} = \Delta_{X_b} - \frac{1}{n}\left(\Delta_{X_b} + \Gamma_{f_b} + \cdots + \Gamma_{f_b^{n-1}}\right) \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$ acts on $H^4(X_b, \mathbf{Q})$ as the orthogonal projector onto $H^4(X_b, \mathbf{Q})^{\#}$ and acts as zero on $H^0(X_b, \mathbf{Q})$, $H^2(X_b, \mathbf{Q})$, $H^6(X_b, \mathbf{Q})$, $H^8(X_b, \mathbf{Q})$. Now we have some control over the cohomology class of 'fibers' of Γ:

Proposition 3.5.2. *For any $b \in B$,*

- (i) *Let $\Gamma_b := \Gamma|_{X_b \times X_b} \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$. Then its cohomology class $[\Gamma_b] \in H^8(X_b \times X_b, \mathbf{Q})$ is contained in $H^4(X_b, \mathbf{Q})^{\#} \otimes H^4(X_b, \mathbf{Q})^{\#}$.*

- (ii) *The cohomology class $[\Gamma_b]$ is supported on $Y_b \times Y_b$, where Y_b is a closed algebraic subset of X_b of codimension at least 2.*

- (iii) *Moreover, there exists an algebraic cycle $\Gamma'_b \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$, which is supported on $Y_b \times Y_b$ and such that $[\Gamma_b] = [\Gamma'_b]$ in $H^8(X_b \times X_b, \mathbf{Q})$*

Proof. (i). By Künneth formula and Poincaré duality, we make the identification

$$H^8(X_b \times X_b) = H^0 \otimes H^8 \oplus H^2 \otimes H^6 \oplus H^6 \otimes H^2 \oplus H^8 \otimes H^0 \oplus H^{4,inv} \otimes H^{4,inv} \oplus H^{4,inv} \otimes H^{4,\#} \oplus H^{4,\#} \otimes H^{4,inv} \oplus H^{4,\#} \otimes H^{4,\#}.$$

By construction, the cohomology class $[\Gamma_b]$ can only have the last component non-zero.
(ii) is a consequence of (iii).
(iii). By Lemma 3.5.1(iv), $H^4(X_b, \mathbf{Q})^{\#}$ is generated by the classes of some codimension 2 subvarieties in X_b. We thus assume $H^4(X_b, \mathbf{Q})^{\#} = \mathbf{Q}[W_1] \oplus \cdots \oplus \mathbf{Q}[W_r]$, where W_i's are subvarieties of codimension 2 in X_b. Let $a_{ij} := (W_i \bullet W_j)$ be the intersection matrix, which is non-degenerate by Poincaré duality. We take $Y_b := \bigcup_{i=1}^r W_i$ to be the codimension 2 closed algebraic subset, and take $\Gamma'_b := \Sigma_{j=1}^r b_{ij} W_i \times W_j \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$, where (b_{ij}) is the inverse matrix of (a_{ij}). It is now easy to check that the cohomological actions of $[\Gamma_b]$ and $[\Gamma'_b]$ are the same, thus $[\Gamma_b] = [\Gamma'_b]$ in $H^8(X_b \times X_b, \mathbf{Q})$. □

Roughly speaking, the previous proposition says that when restricted to each fiber, the cycle Γ becomes homologically equivalent to a cycle supported on a codimension 2 algebraic subset of the fiber. Now here comes the crucial proposition, which allows us to get some global information about Γ from its fiberwise property. The proposition appeared as a key point in Voisin's paper [74]:

Proposition 3.5.3. *In the above situation as in Proposition 3.5.2(iii), there exist a closed algebraic subset $\mathscr{Y}$ in $\mathscr{X}$ of codimension 2, and an algebraic cycle $\Gamma' \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}$ which is supported on $\mathscr{Y} \times_B \mathscr{Y}$, such that for any $b \in B$, $[\Gamma'|_{X_b \times X_b}] = [\Gamma|_{X_b \times X_b}]$ in $H^8(X_b \times X_b, \mathbf{Q})$.*

Proof. See [74] Proposition 2.7. □

Let $\mathscr{Y} \subset \mathscr{X}$ be the codimension 2 closed algebraic subset introduced above, and $\Gamma' \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}$ be the cycle supported on $\mathscr{Y} \times_B \mathscr{Y}$, as constructed in Proposition 3.5.3. Define

$$\mathscr{Z} := \Gamma - \Gamma' \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}. \tag{3.31}$$

Then by construction, for any $b \in B$, the 'fiber' $Z_b := \mathscr{Z}|_{X_b \times X_b} \in \mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$ has trivial cohomology class:

$$[Z_b] = 0 \in H^8(X_b \times X_b, \mathbf{Q}), \qquad \text{for any } \ b \in B. \tag{3.32}$$

The next step is to prove the following decomposition of the projector (Proposition 3.5.4) from the fiberwise cohomological triviality of $\mathscr{Z}$ in (3.32).

Proposition 3.5.4. *There exist a closed algebraic subset $\mathscr{Y}$ in $\mathscr{X}$ of codimension 2, an algebraic cycle $\Gamma' \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}$ supported on $\mathscr{Y} \times_B \mathscr{Y}$, $\mathscr{Z}' \in \mathrm{CH}^4(\mathscr{X} \times \mathbf{P}^5)_{\mathbf{Q}}$ and $\mathscr{Z}'' \in \mathrm{CH}^4(\mathbf{P}^5 \times \mathscr{X})_{\mathbf{Q}}$ such that we have an equality in $\mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}$:*

$$\Delta_{\mathscr{X}} - \frac{1}{n}\left(\Delta_{\mathscr{X}} + \Gamma_f + \cdots + \Gamma_{f^{n-1}}\right) = \Gamma' + \mathscr{Z}'|_{\mathscr{X} \times_B \mathscr{X}} + \mathscr{Z}''|_{\mathscr{X} \times_B \mathscr{X}}. \tag{3.33}$$

To pass from the fiberwise equality (3.32) to the global equality (3.33) above, we have to firstly deduce from (3.32) some global equality up to homological equivalence, then use the result of §3.4 to get an equality up to rational equivalence. The argument of Leray spectral sequence due to Voisin [74, Lemma 2.11] (in our equivariant case) can accomplish the first step.

By Deligne's theorem [29], the Leray spectral sequence associated to the smooth projective morphism $\pi \times \pi : \mathscr{X} \times_B \mathscr{X} \to B$ degenerates at E_2:

$$E_\infty^{p,q} = E_2^{p,q} = H^p(B, R^q(\pi \times \pi)_* \mathbf{Q}) \Rightarrow H^{p+q}\left(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q}\right).$$

In other words,

$$\mathrm{Gr}_L^p H^{p+q}(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q}) = H^p(B, R^q(\pi \times \pi)_* \mathbf{Q}),$$

where $L^\bullet$ is the resulting *Leray filtration* on $H^*(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q})$. The property (3.32) is thus equivalent to

Lemma 3.5.5. *The cohomology class $[\mathscr{Z}] \in L^1 H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q})$.*

Proof. The image of $[\mathscr{Z}] \in H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q})$ in the first graded piece $\mathrm{Gr}_L^0 H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q}) = H^0(B, R^8(\pi \times \pi)_* \mathbf{Q})$ is a section of the vector bundle $R^8(\pi \times \pi)_* \mathbf{Q}$, whose fiber over $b \in B$ is $H^8(X_b \times X_b, \mathbf{Q})$. The value of this section on b is given exactly by $[Z_b] \in H^8(X_b \times X_b, \mathbf{Q})$, which vanishes by (3.32). Therefore $[\mathscr{Z}] \in H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q})$ becomes zero in the quotient Gr_L^0, hence is contained in L^1. □

Consider the Leray spectral sequences associated to the following three smooth projective morphisms to the base B:

$$\begin{array}{ccccc} \mathscr{X} \times \mathbf{P}^5 & \xleftarrow{\ \mathrm{id} \times i\ } & \mathscr{X} \times_B \mathscr{X} & \xrightarrow{\ i \times \mathrm{id}\ } & \mathbf{P}^5 \times \mathscr{X} \\ & \searrow^{\pi \circ \mathrm{pr}_1} & \downarrow^{\pi \times \pi} & \swarrow_{\pi \circ \mathrm{pr}_2} & \\ & & B & & \end{array}$$

and the restriction maps for cohomology induced by the two inclusions. We have the following lemma, where all the cohomology groups are of rational coefficients.

Lemma 3.5.6. *Let $L^\bullet$ be the Leray filtrations corresponding to the above Leray spectral sequences.*

(i) *The Künneth isomorphisms induce canonical isomorphisms*

$$L^1H^8(\mathscr{X}\times\mathbf{P}^5)=\bigoplus_{i+j=8}L^1H^i(\mathscr{X})\otimes H^j(\mathbf{P}^5),$$

$$L^1H^8(\mathbf{P}^5\times\mathscr{X})=\bigoplus_{i+j=8}H^i(\mathbf{P}^5)\otimes L^1H^j(\mathscr{X}).$$

(ii) *The natural restriction map*

$$L^1H^8(\mathscr{X}\times\mathbf{P}^5)\oplus L^1H^8(\mathbf{P}^5\times\mathscr{X})\twoheadrightarrow L^1H^8(\mathscr{X}\times_B\mathscr{X})$$

is surjective.

Proof. By snake lemma (or five lemma) and induction, it suffices to prove the corresponding results for the graded pieces.
(i) We only prove the first isomorphism, the second one is similar. For any $p\geq 1$, the isomorphism

$$\mathrm{Gr}_L^pH^8(\mathscr{X}\times\mathbf{P}^5)=\bigoplus_{i+j=8}\mathrm{Gr}_L^pH^i(\mathscr{X})\otimes H^j(\mathbf{P}^5)$$

by Deligne' theorem is equivalent to

$$H^p(B,R^{8-p}(\pi\circ\mathrm{pr}_1)_*\mathbf{Q})=H^p\Big(B,\bigoplus_{i+j=8}\Big(R^{i-p}\pi_*\mathbf{Q}\Big)\otimes_{\mathbf{Q}}H^j(\mathbf{P}^5)\Big).$$

However, $R^{8-p}(\pi\circ\mathrm{pr}_1)_*\mathbf{Q}$ is a vector bundle with fiber $H^{8-p}(X_b\times\mathbf{P}^5,\mathbf{Q})$, which is by Künneth formula isomorphic to $\oplus_{i+j=8}H^{i-p}(X_b)\otimes H^j(\mathbf{P}^5)$, which is exactly the fiber of the vector bundle $\oplus_{i+j=8}\Big(R^{i-p}\pi_*\mathbf{Q}\Big)\otimes_{\mathbf{Q}}H^j(\mathbf{P}^5)$. Thus ($i$) is a consequence of the relative Künneth formula.
(ii) Using (i), for any $p\geq 1$, the surjectivity of $\mathrm{Gr}_L^pH^8(\mathscr{X}\times\mathbf{P}^5)\oplus\mathrm{Gr}_L^pH^8(\mathbf{P}^5\times\mathscr{X})\twoheadrightarrow\mathrm{Gr}_L^pH^8(\mathscr{X}\times_B\mathscr{X})$ is by Deligne's theorem equivalent to the surjectivity of

$$H^p\Big(B,\bigoplus_{i+j=8}\Big(R^{i-p}\pi_*\mathbf{Q}\Big)\otimes_{\mathbf{Q}}H^j(\mathbf{P}^5)\Big)\oplus H^p\Big(B,\bigoplus_{i+j=8}H^i(\mathbf{P}^5)\otimes_{\mathbf{Q}}\Big(R^{j-p}\pi_*\mathbf{Q}\Big)\Big)\twoheadrightarrow H^p(B,R^{8-p}(\pi\times\pi)_*\mathbf{Q}).$$

By relative Künneth isomorphism, $R^{8-p}(\pi\times\pi)_*\mathbf{Q}=\bigoplus_{k+l=8-p}R^k\pi_*\mathbf{Q}\otimes R^l\pi_*\mathbf{Q}$. Since $8-p\leq 7$, either $k<4$ or $l<4$. Recall that $H^{odd}(X_b)=0$ and the restriction map $H^{2i}(\mathbf{P}^5)\to H^{2i}(X_b)$ is an isomorphism for $i=0,1,3,4$, thus $R^{8-p}(\pi\times\pi)_*\mathbf{Q}$ is a direct summand of the vector bundle $\bigoplus_{k+l=8-p}R^k\pi_*\mathbf{Q}\otimes_{\mathbf{Q}}H^l(\mathbf{P}^5)\oplus\bigoplus_{k+l=8-p}H^k(\mathbf{P}^5)\otimes_{\mathbf{Q}}R^l\pi_*\mathbf{Q}$. Therefore the above displayed morphism is induced by the projection of a vector bundle to its direct summand, which is of course surjective on cohomology. □

Combining Lemma 3.5.5 and Lemma 3.5.6, we can decompose the cohomology class $[\mathscr{Z}]$ as follows:

$$[\mathscr{Z}]=\sum_{i=0}^4\mathrm{pr}_1^*A_i\cdot\mathrm{pr}_2^*[h]^{4-i}+\sum_{j=0}^4\mathrm{pr}_1^*[h]^{4-j}\cdot\mathrm{pr}_2^*B_j\in H^8(\mathscr{X}\times_B\mathscr{X},\mathbf{Q}),\tag{3.34}$$

where $\mathrm{pr}_i:\mathscr{X}\times_B\mathscr{X}\to\mathscr{X}$ is the i-th projection, $A_i\in H^{2i}(\mathscr{X},\mathbf{Q})$, $B_j\in H^{2j}(\mathscr{X},\mathbf{Q})$ and $h\in\mathrm{CH}^1(\mathscr{X})$ is the pull back by the natural morphism $\mathscr{X}\to\mathbf{P}^5$ of the hyperplane divisor $c_1(\mathscr{O}_{\mathbf{P}^5}(1))$.

We remark that A_i and B_j must be *algebraic*, that is, they are the cohomology classes of algebraic cycles of $\mathscr{X}$. The reason is very simple: $[\mathscr{Z}]$ being algebraic, so is

$$\mathrm{pr}_{1,*}\Big([\mathscr{Z}]\cdot\mathrm{pr}_2^*[h]^i\Big)=3A_i+(\text{a rational number})\,[h]^i,$$

thus A_i is algebraic. The algebraicity of B_j is similar. We denote still by $A_i \in \mathrm{CH}^i(\mathscr{X})_{\mathbf{Q}}$ and $B_j \in \mathrm{CH}^j(\mathscr{X})_{\mathbf{Q}}$ for the algebraic cycles with the respective cohomology classes. Therefore (3.34) becomes

$$[\mathscr{Z}] = \sum_{i=0}^{4}[\mathrm{pr}_1^* A_i \cdot \mathrm{pr}_2^* h^{4-i}] + \sum_{j=0}^{4}[\mathrm{pr}_1^* h^{4-j} \cdot \mathrm{pr}_2^* B_j] \in H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q}).$$

In other words, there exist algebraic cycles

$$\mathscr{Z}' := \sum_{i=0}^{4} \mathrm{pr}_1^* A_i \cdot \mathrm{pr}_2^* h^{4-i} \in \mathrm{Im}\left(\mathrm{CH}^4(\mathscr{X} \times \mathbf{P}^5)_{\mathbf{Q}} \to \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}\right), \quad \text{and}$$

$$\mathscr{Z}'' := \sum_{j=0}^{4} \mathrm{pr}_1^* h^{4-j} \cdot \mathrm{pr}_2^* B_j \in \mathrm{Im}\left(\mathrm{CH}^4(\mathbf{P}^5 \times \mathscr{X})_{\mathbf{Q}} \to \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}\right),$$

such that

$$[\mathscr{Z}] = [\mathscr{Z}' + \mathscr{Z}''] \in H^8(\mathscr{X} \times_B \mathscr{X}, \mathbf{Q}). \tag{3.35}$$

This is an equality up to homological equivalence. Now enters the result of §3.4: thanks to Proposition 3.4.1, up to shrinking B to a dense open subset (still denoted by B), the cohomological decomposition (3.35) in fact implies the following equality up to *rational equivalence* in Chow groups:

$$\mathscr{Z} = \mathscr{Z}'|_{\mathscr{X}\times_B\mathscr{X}} + \mathscr{Z}''|_{\mathscr{X}\times_B\mathscr{X}} \in \mathrm{CH}^4(\mathscr{X} \times_B \mathscr{X})_{\mathbf{Q}}.$$

Combining this with (3.28), (3.30), Proposition 3.5.3 and (3.31), we obtain a decomposition of the projector (3.33) announced in Proposition 3.5.4.

Now we can deduce Theorem 3.3.3 from this decomposition as follows. For any $b \in B$ (thus general in $\overline{B}$), taking the fiber of (3.33) over b, we get an equality in $\mathrm{CH}^4(X_b \times X_b)_{\mathbf{Q}}$.

$$\Delta_{X_b} = \frac{1}{n}\left(\Delta_{X_b} + \Gamma_f + \cdots + \Gamma_{f^{n-1}}\right) + \Gamma'_b + Z'_b|_{X_b\times X_b} + Z''_b|_{X_b\times X_b}, \tag{3.36}$$

where we still write f for f_b the restriction of the action on fiber X_b, Γ'_b is supported on $Y_b \times Y_b$ with Y_b a closed algebraic subset of codimension 2 in X_b, and Z'_b (*resp.* Z''_b) is a cycle of $X_b \times \mathbf{P}^5$ (*resp.* $\mathbf{P}^5 \times X_b$) with rational coefficients.

For any homologically trivial 1-cycle $\gamma \in \mathrm{CH}_1(X_b)_{\mathbf{Q},hom}$, let both sides of (3.36) act on it by correspondences. We have in $\mathrm{CH}_1(X_b)_{\mathbf{Q}}$:

- $\Delta_{X_b}^*(\gamma) = \gamma$;
- $\frac{1}{n}\left(\Delta_{X_b} + \Gamma_f + \cdots + \Gamma_{f^{n-1}}\right)^*(\gamma) = \frac{1}{n}\left(\gamma + f^*\gamma + \cdots + (f^*)^{n-1}\gamma\right)$;
- $\Gamma'^*_b(\gamma) = 0$ because the support of Γ'_b has the projection to the first coordinate codimension 2;
- $\left(Z'_b|_{X_b\times X_b}\right)^*(\gamma) = \left(Z''_b|_{X_b\times X_b}\right)^*(\gamma) = 0$, since they both factorizes through $\mathrm{CH}^*(\mathbf{P}^5)_{\mathbf{Q},hom} = 0$.

As a result, we have in $\mathrm{CH}_1(X_b)_{\mathbf{Q}}$,

$$\gamma = \frac{1}{n}\left(\gamma + f^*\gamma + \cdots + (f^*)^{n-1}\gamma\right),$$

where the right hand side is obviously invariant by f^*, hence so is the left hand side. In other words, f^* acts on $\mathrm{CH}_1(X_b)_{\mathbf{Q},hom}$ as identity. Finally, since $H^6(X_b,\mathbf{Q})$ is 1-dimensional with f^* acting trivially, we have for any $\gamma \in \mathrm{CH}_1(X_b)_{\mathbf{Q}}$,

$$\pi^{inv,*}(\gamma) - \gamma \in \mathrm{CH}_1(X_b)_{\mathbf{Q},hom},$$

where $\pi^{inv,*} = \frac{1}{n}\left(\mathrm{id} + f^* + \cdots + (f^*)^{n-1}\right)$. Therefore by what we just obtained,

$$f^*\left(\pi^{inv,*}(\gamma) - \gamma\right) = \pi^{inv,*}(\gamma) - \gamma.$$

As $\pi^{inv,*}(\gamma)$ is obviously f^*-invariant, we have $f^*(\gamma) = \gamma$ in $\mathrm{CH}_1(X_b)_{\mathbf{Q}}$. Theorem 3.3.3, as well as the main Theorem 3.0.5, is proved.

3.6 A remark

In the main Theorem 3.0.5, we assumed that the automorphism of the Fano variety of lines is induced from an automorphism of the cubic fourfold itself. In this final section we want to reformulate this hypothesis.

Proposition 3.6.1. *Let $X \subset \mathbf{P}^5$ be a (smooth) cubic fourfold, and $(F(X),\mathscr{L})$ be its Fano variety of lines equipped with the Plücker polarization induced from the ambiant Grassmannian* $\mathrm{Gr}(\mathbf{P}^1,\mathbf{P}^5)$. *An automorphism ψ of $F(X)$ comes from an automorphism of X if and only if it is polarized (i.e. $\psi^*\mathscr{L} \simeq \mathscr{L}$).*

Proof. The following proof is taken from [19, Proposition 4]. Consider the projective embedding of $F(X)$ determined by the Plücker polarization $\mathscr{L}$:

$$F(X) =: F \subset \mathrm{Gr}(\mathbf{P}^1,\mathbf{P}^5) =: G \subset \mathbf{P}(\wedge^2 H^0(\mathbf{P}^5,\mathscr{O}(1))^\vee) =: \mathbf{P}^{14}.$$

If ψ is induced from an automorphism f of X, which must be an automorphism of $\mathbf{P}^5$, then it is clear that ψ is the restriction of a linear automorphism of $\mathbf{P}^{14}$, thus the Plücker polarization is preserved.

Conversely, if the automorphism ψ preserves the polarization, it is then the restriction of a projective automorphism of $\mathbf{P}^{14}$, which we denote still by ψ. It is proved in [4, 1.16(iii)] that G is the intersection of all the quadrics containing F. It follows that ψ is an automorphism of G, because ψ sends any quadric containing F to a quadric containing F. However any automorphism of G is induced by a projective automorphism f of $\mathbf{P}^5$ (*cf.* [23]). As a result, f sends a line contained in X to a line contained in X, thus f preserves X and ψ is induced from f. □

Define $\mathrm{Aut}^{pol}(F(X))$ to be the group of polarized automorphisms of $F(X)$ (equipped with Plücker polarization $\mathscr{L}$). Then the previous proposition says that the image of the injective homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(F(X))$ is exactly $\mathrm{Aut}^{pol}(F(X))$, that is:

Corollary 3.6.2. *We have an isomorphism*

$$\begin{array}{ccc} \mathrm{Aut}(X) & \xrightarrow{\simeq} & \mathrm{Aut}^{pol}(F(X)) \\ f & \mapsto & \hat{f} \end{array}$$

Corollary 3.6.2 then gives us the last statement in main Theorem 3.0.5: *any polarized symplectic automorphism of $F(X)$ acts as identity on* $\mathrm{CH}_0(F(X))$.

Remark 3.6.3. By the above isomorphism and Remark 3.1.1, $\mathrm{Aut}(X) \simeq \mathrm{Aut}^{pol}(F(X))$ is a finite group. However, the isomorphism being established, we can show the finiteness without calculation: as before, $\mathrm{Aut}^{pol}(F(X))$ being a closed subgroup of PGL_{15} thus is of finite type, and thanks to the symplectic structure we have

$$H^0(F(X), T_{F(X)}) = H^{1,0}(F(X)) = 0,$$

which implies that $\mathrm{Aut}^{pol}(F(X))$ is discrete, and therefore finite.

3.7 A consequence: action on $\mathrm{CH}_2(F)_{\mathbf{Q},hom}$

As an application of Theorem 3.0.5, we study in this section the induced action on the Chow group of 2-cycles. The conclusion is Corollary 3.0.7 in the introduction.

Proof of Corollary 3.0.7. As showed in the previous section (Corollary 3.6.2), the polarized automorphism $\hat{f}$ on $F(X)$ comes from an automorphism f of finite order n of the smooth cubic fourfold X. We consider again the projector $\Gamma := \Delta_F - \pi^{inv} \in \mathrm{CH}_4(F \times F)_{\mathbf{Q}}$, where

$$\pi^{inv} = \frac{\Delta_F + \Gamma_{\hat{f}} + \cdots + \Gamma_{\hat{f}^{n-1}}}{n} \in \mathrm{CH}_4(F \times F)_{\mathbf{Q}}.$$

We remark that $\Gamma = {}^t\Gamma$ since $\hat{f}^{-1} = \hat{f}^{n-1}$. Our main result Theorem 3.0.5 says in particular that the action of Γ on $\mathrm{CH}_0(F)_{\mathbf{Q}}$ is zero:

$$\Gamma_* = 0 : \mathrm{CH}_0(F)_{\mathbf{Q}} \to \mathrm{CH}_0(F)_{\mathbf{Q}}.$$

Equivalently speaking, the restriction of Γ to each fiber is zero:

$$\Gamma|_{\{t\}\times F} = 0 \in \mathrm{CH}_0(F)_{\mathbf{Q}}, \quad \forall t \in F.$$

By the argument of Bloch-Srinivas (*cf.* [18], [68, §10.2]), there exist an effective reduced divisor $D \subsetneq X$, a resolution of singularities $\tau : \widetilde{D} \to D$ and an algebraic cycle $\Gamma' \in \mathrm{CH}_4(\widetilde{D} \times F)_{\mathbf{Q}}$ such that $\Gamma = (\widetilde{\tau} \times \mathrm{id}_F)_* \Gamma'$, where $\widetilde{\tau}$ is the composition of τ and the inclusion of D into X. Consequently, the action of $\Gamma = {}^t\Gamma$ on $\mathrm{CH}_2(F)_{\mathbf{Q}}$ factorises as:

$$\begin{array}{ccc} \mathrm{CH}_2(F)_{\mathbf{Q}} & \xrightarrow{\Gamma_*=\Gamma^*} & \mathrm{CH}_2(F)_{\mathbf{Q}} \\ & {\scriptstyle \Gamma'^*} \searrow \quad \nearrow {\scriptstyle \widetilde{\tau}_*} & \\ & \mathrm{CH}^1(\widetilde{D})_{\mathbf{Q}} & \end{array}$$

Since these correspondences preserve the homological equivalence as well as the Abel-Jacobi equivalence (*cf.* [68, Chapter 9]), we have in fact the following factorization:

$$\begin{array}{ccc} \mathrm{CH}_2(F)_{\mathbf{Q},AJ} & \xrightarrow{\Gamma_*=\Gamma^*} & \mathrm{CH}_2(F)_{\mathbf{Q},AJ} \\ & {\scriptstyle \Gamma'^*} \searrow \quad \nearrow {\scriptstyle \widetilde{\tau}_*} & \\ & \mathrm{CH}^1(\widetilde{D})_{\mathbf{Q},AJ} & \end{array}$$

where AJ means the Abel-Jacobi kernels. However, it is well-known that for divisors the Abel-Jacobi map is an isomorphism. Hence

$$\mathrm{CH}^1(\widetilde{D})_{\mathbf{Q},AJ} = 0.$$

Now the factorization implies that Γ acts as zero on $\mathrm{CH}_2(F)_{\mathbf{Q},AJ}$, thus $\hat{f}$ acts as identity on $\mathrm{CH}_2(F)_{\mathbf{Q},AJ}$. To conclude, it suffices to remark that from the vanishing $H^3(F) = 0$, the Abel-Jacobi kernel

$$\mathrm{CH}_2(F)_{\mathbf{Q},AJ} := \ker\left(\mathrm{CH}_2(F)_{\mathbf{Q},hom} \to J^3(F) := \frac{H^{0,3}(F) \oplus H^{1,2}(F)}{H^3(F, \mathbf{Z})} = 0\right)$$

is equal to $\mathrm{CH}_2(F)_{\mathbf{Q},hom}$. □

Remark 3.7.1. We want to remark that Corollary 3.0.7 is also predicted by Bloch-Beilinson conjecture (a more general version than Conjecture 3.0.4, *cf.* [68, Chapter 11], [6, Chapitre 11]).

Bibliography

[1] Salman Abdulali, *Abelian varieties and the general Hodge conjecture*, Compositio Math. **109** (1997), no. 3, 341–355. MR 1485922 (98m:14008)

[2] ______, *Hodge structures on abelian varieties of CM-type*, J. Reine Angew. Math. **534** (2001), 33–39. MR 1831630 (2002c:14019)

[3] ______, *Hodge structures on abelian varieties of type III*, Ann. of Math. (2) **155** (2002), no. 3, 915–928. MR 1923969 (2003g:14008)

[4] Allen B. Altman and Steven L. Kleiman, *Foundations of the theory of Fano schemes*, Compositio Math. **34** (1977), no. 1, 3–47. MR 0569043 (58 #27967)

[5] Yves André, *Pour une théorie inconditionnelle des motifs*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 5–49. MR 1423019 (98m:14022)

[6] ______, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004. MR 2115000 (2005k:14041)

[7] Michael Atiyah and Friedrich Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45. MR 0145560 (26 #3091)

[8] Fabio Bardelli, *On Grothendieck's generalized Hodge conjecture for a family of threefolds with trivial canonical bundle*, J. Reine Angew. Math. **422** (1991), 165–200. MR 1133322 (92i:14005)

[9] Arnaud Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782 (1984). MR 730926 (86c:32030)

[10] ______, *On the splitting of the Bloch-Beilinson filtration*, Algebraic cycles and motives. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 344, Cambridge Univ. Press, Cambridge, 2007, pp. 38–53. MR 2187148 (2009c:14007)

[11] Arnaud Beauville and Ron Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706. MR 818549 (87c:14047)

[12] Arnaud Beauville and Claire Voisin, *On the Chow ring of a $K3$ surface*, J. Algebraic Geom. **13** (2004), no. 3, 417–426. MR 2047674 (2005b:14011)

[13] Aleksandr A. Beĭlinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25. MR 923131 (89h:11027)

[14] Marcel Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330. MR 0079806 (18,149a)

[15] Spencer Bloch, *Lectures on algebraic cycles*, second ed., New Mathematical Monographs, vol. 16, Cambridge University Press, Cambridge, 2010. MR 2723320

[16] Spencer Bloch, Arnold Kas, and David Lieberman, *Zero cycles on surfaces with p_g = 0*, Compositio Math. **33** (1976), no. 2, 135–145. MR 0435073 (55 #8035)

[17] Spencer Bloch and Arthur Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975). MR 0412191 (54 #318)

[18] Spencer Bloch and Vasudevan Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), no. 5, 1235–1253. MR 714776 (85i:14002)

[19] François Charles, *A remark on the Torelli theorem for cubic fourfolds*, preprint, arXiv:1209.4509.

[20] François Charles and Eyal Markman, *The Standard Conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces*, To appear in Compositio Mathematica (2011).

[21] Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. MR 0303460 (46 #2597)

[22] Xi Chen, *Rational curves on $K3$ surfaces*, J. Algebraic Geom. **8** (1999), no. 2, 245–278. MR 1675158 (2000d:14057)

[23] Wei-Liang Chow, *On the geometry of algebraic homogeneous spaces*, Ann. of Math. (2) **50** (1949), 32–67. MR 0028057 (10,396d)

[24] Jean-Louis Colliot-Thélène and Claire Voisin, *Cohomologie non Ramifiée et conjecture de Hodge entière*, Duke Math. J. **161** (2012), no. 5, 735–801. MR 2904092

[25] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)

[26] Olivier Debarre, *Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 14, 681–684. MR 770463 (87c:14039)

[27] Olivier Debarre and Laurent Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète.*, Math. Ann. **312** (1998), no. 3, 549–574 (French).

[28] Olivier Debarre and Claire Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. **649** (2010), 63–87. MR 2746467

[29] Pierre Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 259–278. MR 0244265 (39 #5582)

[30] ______, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 0498551 (58 #16653a)

[31] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR 654325 (84m:14046)

[32] Hélène Esnault, Marc Levine, and Eckart Viehweg, *Chow groups of projective varieties of very small degree*, Duke Math. J. **87** (1997), no. 1, 29–58. MR 1440062 (98d:14002)

[33] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037 (94g:14028)

[34] ______, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

[35] Alexander Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188. MR 0244271 (39 #5586c)

[36] ______, *Hodge's general conjecture is false for trivial reasons*, Topology **8** (1969), 299–303. MR 0252404 (40 #5624)

[37] ______, *Standard conjectures on algebraic cycles*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 193–199. MR 0268189 (42 #3088)

[38] Fabien Herbaut, *Algebraic cycles on the Jacobian of a curve with a linear system of given dimension*, Compos. Math. **143** (2007), no. 4, 883–899. MR 2339831 (2008m:14053)

[39] Daniel Huybrechts, *Symplectic automorphisms of K3 surfaces of arbitrary order*, To appear in Math. Res. Letters, arXiv:1205.3433.

[40] ______, *A global torelli theorem for hyperkähler manifolds (after verbitsky)*, Seminaire Bourbaki (2011), arXiv:1111.1745v1.

[41] Daniel Huybrechts and Michael Kemeny, *Stable maps and Chow groups*, preprint, arXiv:1202.4968.

[42] Jun ichi Igusa, *Theta functions*, Springer-Verlag, New York, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 194. MR 0325625 (48 #3972)

[43] Atanas Iliev and Kristian Ranestad, *$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1455–1468 (electronic). MR 1806733 (2002c:14082)

[44] ______, *Addendum to "$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds" [Trans. Amer. Math. Soc.* **353** *(2001), no. 4, 1455–1468; mr1806733]*, C. R. Acad. Bulgare Sci. **60** (2007), no. 12, 1265–1270. MR 2391437 (2009a:14066)

[45] Uwe Jannsen, *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 245–302. MR 1265533 (95c:14006)

[46] Shun-Ichi Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), no. 1, 173–201. MR 2107443 (2005j:14007)

[47] Steven L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix esposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 359–386. MR 0292838 (45 #1920)

[48] ———, *The standard conjectures*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 3–20. MR 1265519 (95k:14010)

[49] Robert Laterveer, *Algebraic varieties with small Chow groups*, J. Math. Kyoto Univ. **38** (1998), no. 4, 673–694. MR 1669995 (99k:14013)

[50] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten, *Twisted cubics on cubic fourfolds*, preprint, arXiv:1305.0178.

[51] Valeria Ornella Marcucci and Gian Pietro Pirola, *Points of order two on theta divisors*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **23** (2012), no. 3, 319–323. MR 2960840

[52] Shigefumi Mori and Shigeru Mukai, *The uniruledness of the moduli space of curves of genus* 11, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 334–353 (English). MR 726433 (85b:14033)

[53] David Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354. MR 0204427 (34 #4269)

[54] ———, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968), 195–204. MR 0249428 (40 #2673)

[55] Jakob P. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples*, Indag. Math. (N.S.) **4** (1993), no. 2, 177–188. MR 1225267 (94j:14006a)

[56] Kieran G. O'Grady, *Desingularized moduli spaces of sheaves on a $K3$*, J. Reine Angew. Math. **512** (1999), 49–117. MR 1703077 (2000f:14066)

[57] ———, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505. MR 1966024 (2004c:14017)

[58] ———, *Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics*, Duke Math. J. **134** (2006), no. 1, 99–137. MR 2239344 (2007e:14062)

[59] Kapil H. Paranjape, *Cohomological and cycle-theoretic connectivity*, Ann. of Math. (2) **139** (1994), no. 3, 641–660. MR 1283872 (95g:14008)

[60] Tetsuji Shioda, *What is known about the Hodge conjecture?*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 55–68. MR 715646 (84k:14007)

[61] Christophe Soulé and Claire Voisin, *Torsion cohomology classes and algebraic cycles on complex projective manifolds*, Adv. Math. **198** (2005), no. 1, 107–127. MR 2183252 (2006i:14006)

[62] Joseph H. M. Steenbrink, *Some remarks about the Hodge conjecture*, Hodge theory (Sant Cugat, 1985), Lecture Notes in Math., vol. 1246, Springer, Berlin, 1987, pp. 165–175. MR 894051 (88f:14008)

[63] Burt Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. **10** (1997), no. 2, 467–493. MR 1423033 (98a:14012)

[64] Claire Voisin, *Bloch's conjecture for Catanese and Barlow surfaces*, preprint, arXiv:1210.3935.

[65] ______, *Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), no. 4, 473–492. MR 1205880 (93m:14005)

[66] ______, *A counterexample to the Hodge conjecture extended to Kähler varieties*, Int. Math. Res. Not. (2002), no. 20, 1057–1075. MR 1902630 (2003e:32037)

[67] ______, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002, Translated from the French original by Leila Schneps. MR 1967689 (2004d:32020)

[68] ______, *Hodge theory and complex algebraic geometry. II*, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2003, Translated from the French by Leila Schneps. MR 1997577 (2005c:32024b)

[69] ______, *Remarks on filtrations on Chow groups and the Bloch conjecture*, Ann. Mat. Pura Appl. (4) **183** (2004), no. 3, 421–438. MR 2082666 (2005e:14009)

[70] ______, *On integral Hodge classes on uniruled or Calabi-Yau threefolds*, Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 43–73. MR 2306166 (2008f:14057)

[71] ______, *On the Chow ring of certain algebraic hyper-Kähler manifolds*, Pure Appl. Math. Q. **4** (2008), no. 3, part 2, 613–649. MR 2435839 (2009m:14004)

[72] ______, *Coniveau 2 complete intersections and effective cones*, Geom. Funct. Anal. **19** (2010), no. 5, 1494–1513. MR 2585582 (2011a:14014)

[73] ______, *Chow rings, decomposition of the diagonal and the topology of families*, Notes of Hermann Weyl lectures. IAS (2011).

[74] ______, *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections*, To appear in Ann. Sci. de l'ENS, arXiv:1107.2600 (2011).

[75] ______, *Chow rings and decomposition theorems for families of $K3$ surfaces and Calabi-Yau hypersurfaces*, Geom. Topol. **16** (2012), no. 1, 433–473. MR 2916291

[76] ______, *Symplectic involutions of $K3$ surfaces act trivially on CH_0*, Documenta Math. **17** (2012), 851–860.

[77] Shing Tung Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411. MR 480350 (81d:53045)

[78] Steven Zucker, *The Hodge conjecture for cubic fourfolds*, Compositio Math. **34** (1977), no. 2, 199–209. MR 0453741 (56 #12001)